Stabilité d'ondes périodiques, schéma numérique pour le chimiotactisme

Valérie Le Blanc

N° d'ordre : 2010 - 91

Université de Lyon

Université Claude Bernard Lyon 1

Institut Camille Jordan

École doctorale InfoMaths

THÈSE

Spécialité mathématiques

en vue d'obtenir le grade de docteur,
présentée et soutenue publiquement par

Mme Valérie Le Blanc

le 24 juin 2010.

Stabilité d'ondes périodiques, Schéma numérique pour le chimiotactisme

---

Thèse soutenue devant la commission d'examen formée de

Mme Sylvie Benzoni-Gavage	*membre*
M. Francis Filbet	*directeur*
M. Thierry Goudon	*membre*
M. Philippe Laurençot	*rapporteur*
M. Pascal Noble	*directeur*
M. Frédéric Rousset	*rapporteur*

Résumé

Cette thèse est articulée autour de deux facettes de l'étude des équations aux dérivées partielles. Dans une première partie, on étudie la stabilité des solutions périodiques pour des lois de conservation. On démontre d'abord la stabilité asymptotique dans L^1 des solutions périodiques de lois de conservation scalaires et inhomogènes. On montre ensuite un résultat de stabilité structurelle des *roll-waves*. Plus précisément, on montre que les solutions périodiques d'un système hyperbolique sans viscosité sont limites des solutions du problème avec viscosité, quand le terme de viscosité tend vers 0. Dans une deuxième partie, on s'intéresse à un système d'équations aux dérivées partielles issu de la biologie : le modèle de Patlak-Keller-Segel en dimension 2 ; il décrit les phénomènes de chimiotactisme. Pour ce modèle, on construit un schéma de type volume fini, ce qui permet d'approcher la solution tout en gardant certaines propriétés du système : positivité, conservation de la masse, estimation d'énergie.

Abstract

This thesis is organized around two aspects of the study of partial differential equations. In a first part, we study the stability of periodic solutions for conservation laws. First, we prove asymptotic L^1-stability of periodic solutions of scalar inhomogeneous conservation laws. Then, we show a result on structural stability of roll-waves. More precisely, we prove that periodic solutions of a hyperbolic system without viscosity are the limits of the solutions of the problem with viscosity, as the viscous term tends to 0. In a second part, we study a system of partial differential equations derived from biology: the model of Patlak-Keller-Segel in dimension 2, describing the phenomena of chemotaxis. For this model, we construct a finite-volume scheme, which approaches the solution while keeping some properties of the system: positivity, conservation of mass, energy estimate.

Remerciements

Avant toute chose, je tiens à remercier mes directeurs de thèse Francis Filbet et Pascal Noble pour m'avoir guidée dans mes recherches : tant en me proposant des problèmes intéressants, qu'en étant là pour m'aider à avancer, répondre à mes questions et me relire.

Je suis également très reconnaissante envers mes rapporteurs Philippe Laurençot et Frédéric Rousset pour leur lecture attentive de ma thèse, ce qui m'a permis d'y apporter quelques améliorations. Je remercie Sylvie Benzoni-Gavage et Thierry Goudon d'avoir accepté de faire partie de mon jury.

Je profite de cette page pour remercier l'ICJ et ses membres pour leur accueil et leur convivialité. En particulier, Monique Gaffier qui est toujours disponible pour aider dans les tâches administratives.

Je remercie également l'UMPA et ses membres pour m'avoir donné un bureau pendant toute la durée de ma thèse, ce qui m'a permis d'être présente auprès de mes étudiants. J'y ai en particulier beaucoup apprécié le contact entre doctorants, membres permanents et étudiants.

Un grand merci à toutes les personnes avec lesquelles j'ai parlé de sujets mathématiques durant ces années. Je pense en particulier à mes co-bureaux Paul Laurain, Amélie Rambaud, Élodie Orquera-Pozzi, Jean Lécureux, Émeric Bouin, mais également à Magali Mercier et Séverine Enault qui ont toujours été disponibles pour m'écouter raconter mes mathématiques et pour enrichir mes connaissances en me racontant leurs recherches. Je pense enfin à Vincent Calvez qui a répondu à mes questions et s'est intéressé à mes avancées.

Cette page ne serait pas complète sans un sincère remerciement aux personnes qui m'ont accompagnée pendant ces années et dont l'amitié est importante à mes yeux. Commençons par le « collectif rue Pasteur » : Béné, Benji, Isa, Lucile, Magali, Marianne, Marie, Séverine avec qui j'ai partagé beaucoup de bons moments et de blagues, et qui ont supporté ma mauvaise humeur chronique. Citons également la chorale avec une pensée particulière pour certains membres : Anne-Cécile, Corentin, Gaëlle, JF et Malice qui ont tous été très présents. Pour finir, je remercie tous les doctorants de l'ICJ et de l'UMPA avec lesquels j'ai eu l'occasion de partager des moments de travail et de pause.

Je tiens également à remercier tous ceux qui sont venues à ma soutenance.

Ne pouvant être exhaustive, j'ai oublié de remercier un certain nombre de personnes, je prie celles-ci de m'en excuser.

Table des matières

Table des figures

Introduction

Les équations aux dérivées partielles modélisent des phénomènes physiques ou biologiques dans lesquels peuvent apparaître des structures : phénomènes cycliques, vagues, agrégats de matière,... Dans cette thèse, on s'intéresse d'une part à l'étude théorique autour des phénomènes périodiques (qui peuvent s'apparenter aux vagues) et d'autre part à la mise en place d'un schéma numérique pour le chimiotactisme, qui concerne des mouvements de bactéries.

Les équations de Saint Venant, qui modélisent des écoulements de fluide en faible profondeur, ont des solutions particulières qui sont des ondes progressives périodiques, appelées *roll-waves*. Ce type de structures périodiques, trouvé pour Saint Venant par R. F. Dressler [24], apparaît également dans d'autres équations. L'existence de *roll-waves* de petite amplitude a notamment été généralisée à des systèmes hyperboliques généraux par P. Noble [54]. A.-L. Dalibard a par ailleurs montré l'existence de solutions stationnaires périodiques pour des équations de convection-diffusion scalaires non homogènes [19].

Ces structures périodiques appellent un certain nombre de questions sur leur stabilité, tant asymptotique que structurelle.

L'étude de la stabilité asymptotique des ondes périodiques a débuté avec le travail de R. A. Gardner sur la stabilité spectrale pour des systèmes de réaction-diffusion [30]. M. Oh et K. Zumbrun ont ensuite poursuivi ces travaux dans le cadre des lois de conservations visqueuses : un développement asymptotique de la fonction de Evans permet d'obtenir des conditions nécessaires pour avoir de la stabilité spectrale [57]. Ils ont également montré par des estimations sur les fonctions de Green que, sous des hypothèses fortes de stabilité spectrale, on obtient de la stabilité linéaire asymptotique de $L^1 \cap L^p \to L^p$ pour $p > 1$ [58, 56].

Dans le premier chapitre de cette thèse, on s'intéresse à la stabilité asymptotique non linéaire de solutions périodiques d'équations scalaires de la forme :

$$\begin{cases} \partial_t u + \operatorname{div}(f(u,x)) = \Delta u, & x \in \mathbb{R}^d, t > 0, \\ u(\cdot,0) \equiv u_0, & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

où f est une fonction lisse supposée périodique par rapport à la variable x sur un réseau Y de $\mathbb{R}^d$. Compte tenu du fait que les équations de conservation scalaires présentent de la contraction dans L^1 :

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u - v\|_{L^1} \leqslant 0,$$

il est naturel de chercher de la stabilité $L^1 \to L^1$.

Plus précisément, on montre que les solutions stationnaires périodiques proposées par A.-L. Dalibard [19] sont stables dans L^q pour $q > 1$ et d quelconque et qu'elles sont stables dans L^1 pour $d = 1$. En effet, Dalibard donne une famille de solutions stationnaires Y-périodiques w_p paramétrées par leurs moyennes sur Y : p. On considère alors comme donnée initiale une perturbation de w_p dans

$$L^1_0(\mathbb{R}^d) = \left\{ u \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 0 \right\},$$

$u_0 = w_p + b, b \in L^1_0(\mathbb{R}^d)$. On obtient le théorème suivant dans le cas de la dimension 1 :

Théorème 1. (voir p. 17) *Pour tout $p \in \mathbb{R}, b \in L^1_0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, l'unique solution u dans $L^\infty_{loc}(\mathbb{R}^+, L^\infty(\mathbb{R}))$ de (1) avec $u_0 = w_p + b$ vérifie :*

$$\lim_{t\to\infty} \|u(t,\cdot) - w_p\|_1 = 0.$$

De plus, quelle que soit la dimension d'espace, on obtient de la stabilité asymptotique dans L^2 et donc, en interpolant, dans $L^q, q > 1$.

Proposition 1. (voir p. 23) *Soit $R \in \mathbb{R}$. Alors, il existe $C > 0$ tel que pour tout $p \in \mathbb{R}, b \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ qui vérifient l'encadrement $w_{-R} \leqslant w_p + b \leqslant w_R$, la solution u de (1) converge vers w_p dans L^2*

$$\|u(t) - w_p\|_2 \leqslant C_d \frac{\|b\|_1}{t^{d/4}}. \tag{2}$$

Pour démontrer ces résultats, on s'inspire de ce qui a été fait dans le cas où f ne dépend pas de x [28, 65], c'est-à-dire dans le cas où les solutions stationnaires sont les constantes.

Théorème 2. [65] *Pour tout* $k \in \mathbb{R}, b \in L^1_0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$*, l'unique solution* $u \in L^\infty_{loc}(\mathbb{R}^+, L^\infty(\mathbb{R}^d))$ *de*

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = \Delta u, & x \in \mathbb{R}^d, t > 0, \\ u(0,\cdot) \equiv u_0, & x \in \mathbb{R}^d, \end{cases} \tag{3}$$

avec la condition initiale

$$u_0 = k + b,$$

satisfait la convergence

$$\lim_{t\to\infty} \|u(t,\cdot) - k\|_1 = 0.$$

Selon la dimension d'espace, ($d = 1$ ou $d \geqslant 1$), deux méthodes différentes ont été développées. La première, élaborée par H. Freistühler et D. Serre [28] utilise des méthodes et résultats propres à la dimension 1 : intégration de l'équation sur u, estimation du nombre de changements de sens de variation de la solution d'un problème parabolique. La seconde, développée par D. Serre [65], est plus générale et utilise en fait de la stabilité linéaire dans L^1 pour l'équation de la chaleur. En effet, par des changements de variable et d'inconnue, on peut voir le terme div $f(u)$ comme une perturbation quadratique sur l'équation de la chaleur, dont le noyau,

$$K^t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x\|^2}{4t}\right),$$

vérifie en particulier les estimations pour tout a dans $L^1(\mathbb{R}^d)$:

$$\|K^t * a\|_{L^1} \leqslant \|a\|_{L^1},$$
$$\|\nabla K^t * a\|_{L^1} \leqslant C\frac{1}{\sqrt{t}}\|a\|_{L^1}.$$

Malgré leurs différences, les deux méthodes ont un point commun : elles utilisent également de la stabilité dans L^2. Celle-ci vient du fait que $u \to u^2$ est une entropie strictement convexe de l'équation autonome (3). Plus généralement, vu que u est scalaire, toute fonction convexe η est une entropie pour (3) : il existe bien g tel que $g'(u) = \eta'(u)f'(u)$. Les fonctions $u \to (u-k)^+$ pour $k \in \mathbb{R}$, en particulier, sont des entropies : elles s'apparentent aux entropies de Kružkov. Dans le cas de l'équation non autonome (1), les fonctions convexes ne sont pas toutes des entropies. Il a donc fallu trouver une entropie strictement convexe qui nous permette d'obtenir une inégalité de dispersion (2). Cette entropie est construite en utilisant un analogue aux entropies de Kružkov où les constantes sont remplacées par les solutions stationnaires périodiques $u \to (u - w_p)^+, p \in \mathbb{R}$, et en intégrant ces entropies par rapport à p, de sorte à obtenir une fonction strictement convexe par rapport à u. Les calculs nous donnent alors de la convergence dans L^2 de u vers w_p sous la seule condition que $u_0 - w_p$ soit dans L^1 (la condition de masse nulle n'est pas nécessaire ici).

Pour conclure sur la stabilité L^1 dans le cas de la dimension 1, on utilise, comme dans [28], l'équation vérifiée par une primitive V de u, des estimations dans L^2 sur V ainsi qu'un lemme de H. Matano [49] sur le nombre de changements de sens de variation de V. Dans le cas de la dimension d quelconque, un changement de variable et d'inconnue ne suffit plus pour se ramener à une perturbation quadratique de l'équation de la chaleur. Par conséquent, on a besoin pour conclure d'avoir de la stabilité linéaire $L^1 \to L^1$ sur un opérateur de convection-diffusion qui soit le linéarisé de (1). Donnant uniquement de la stabilité dans L^1 en temps court, les résultats de M. Oh et K. Zumbrun ne permettent donc pas encore de conclure sur les questions de stabilité en temps long. [56].

Ceci étant, A.-L. Dalibard a depuis montré qu'on a bien de la stabilité dans L^1 pour toute dimension d'espace [20]. Pour cela, elle utilise un développement asymptotique de la solution par rapport à la période des oscillations et elle étudie précisément les modulations basses fréquences. Obtenant ainsi des bornes dans des espaces L^2 à poids, elle conclut par compacité dans L^1.

Le deuxième chapitre de cette thèse est orienté vers la stabilité structurelle des ondes périodiques. Tandis que P. Noble s'est intéressé à la persistance des *roll-waves* pour l'équation de Saint Venant hyperbolique [53], on s'intéresse ici à avoir de la persistance avec l'ajout de viscosité. Le cas le plus simple où cette persistance a lieu est peut-être celui des profils de choc visqueux pour les équations scalaires en une dimension d'espace. En effet, si on considère la loi de conservation

$$\partial_t u + \partial_x f(u) = 0, \qquad x \in \mathbb{R}, t > 0,$$

et des états $u^- \neq u^+$ qui vérifient la condition de Rankine-Hugoniot ($f(u^+) - f(u^-) = s(u^+ - u^-)$) et la condition de choc de Lax ($f'(u^+) < s < f'(u^-)$), $s \in \mathbb{R}$, alors le système visqueux qui en découle

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u, \qquad x \in \mathbb{R}, t > 0,$$

admet des solutions qui connectent u^- à u^+. De plus, sous une hypothèse sur les positions relatives du graphe de f et de la corde, ces solutions sont des profils de choc : elles s'écrivent donc sous la forme $u^\varepsilon(t,x) = U\left(\frac{x-st}{\varepsilon}\right)$ et tendent vers la solution du problème non visqueux quand ε tend vers 0. Il a ainsi été montré que la structure est conservée en ajoutant de la viscosité, qui apporte de la régularité.

Plus généralement, on considère le système strictement hyperbolique en une dimension d'espace :

$$\partial_t u + \partial_x f(u) = g(u), \qquad x \in \mathbb{R}, t > 0. \tag{4}$$

Le postulat, classique, qui nous intéresse est que tout u solution de l'équation non visqueuse (4) est la limite quand ε tend vers 0 des solutions u^ε de l'équation avec viscosité

$$\partial_t u + \partial_x f(u) = g(u) + \varepsilon \partial_x^2 u, \qquad x \in \mathbb{R}, t > 0, \tag{5}$$

avec la même donnée initiale. Ce résultat, connu pour les lois de conservation scalaires [71] et pour certains systèmes 2×2 [23], s'avère être plus difficile pour les systèmes généraux. Le cas d'un nombre fini de chocs a déjà été traité par J. Goodman et Z. P. Xin [32] dans le cas de chocs de petites amplitudes, d'une part, et par F. Rousset [63] pour une amplitude quelconque, d'autre part. Enfin, dans le cas de petite variation totale, A. Bressan et T. Yang donnent même une estimation de la vitesse de convergence [12].

Comme dit précédemment, on s'intéresse ici à la persistance des solutions périodiques de (4), ou de perturbation de ces dernières. Par conséquent, on traite un cas où les solutions de (4) ont une infinité de chocs. Cependant, pour des raisons qui semblent uniquement techniques, on a besoin d'une certaine structure sur u, solution de (4) : elle doit être périodique, avec une période aussi grande que l'on veut et un nombre fini de chocs par période. Par exemple, u peut être une perturbation périodique d'une *roll-wave*, dans le cas des équations de Saint Venant ou de systèmes hyperboliques plus généraux, perturbation dont l'existence a été montrée par P. Noble [53, 54].

Pour démontrer le résultat de persistance (que l'on énonce plus bas), on s'inspire largement des méthodes utilisées par F. Rousset [63] : construction d'une solution approchée pour (5), construction itérative de la fonction de Green du linéarisé de (5) autour de la solution approchée pour trouver des estimations. La difficulté ici est, d'une part, de prendre en compte la présence de tous les chocs et de les fixer simultanément, et, d'autre part, d'assurer que l'on recolle bien la solution approchée, de sorte à garder la périodicité que l'on demande au départ.

Donnons à présent les hypothèses dont nous avons besoin, ainsi que le résultat principal. En ce qui concerne les hypothèses :

(H1) Le système (4) est strictement hyperbolique.

(H2) u est solution entropique de (4) sur $[0; T^*], T^* > 0$. De plus, u est une fonction lisse par morceaux, L-périodique et a m chocs de Lax par période, dont on suppose qu'ils ne se croisent pas.

En particulier, cela signifie que u est lisse en dehors des courbes de choc lisses $x = X_j(t) + iL, j = 1, \dots, m, i \in \mathbb{Z}$ et que pour tout $j, k, t, |X_j(t) - X_k(t)| > 0$.

(H2') Pour chaque choc, il existe un profil de choc visqueux, pour $t \in [0; T^*]$.

Cela signifie que pour tout j, t, il existe $V^j(t)$ tel que

$$\partial_\xi^2 V^j - \partial_\xi (f(V^j) - X_j' V^j) = 0 \tag{6}$$

et

$$V^j(\pm\infty, t) = \lim_{x \to X_j(t)\pm} u(t, x).$$

(H3) Considérons pour $\tau \leqslant T^*$ l'opérateur

$$\mathcal{L}_\tau^j w = \partial_z^2 w - (df(V^j(z,\tau)) - X_j'(\tau))\partial_z w.$$

On suppose que le profil de choc visqueux est linéairement stable pour cet opérateur.

Sous ces hypothèses, on montre le théorème suivant

Théorème 3. (voir p. 36) *Sous les hypothèses* **(H1)–(H2)–(H2')–(H3)**, *et si g est linéaire ($g(u) = \kappa u$), alors pour tout $\varepsilon > 0$, il existe une solution u^ε L-périodique de (5) sur $[0; T^*]$ telle que*

$$\|u^\varepsilon(t=0) - u(t=0)\|_{L^1(0;L)} = 0, \qquad \textit{quand } \varepsilon \to 0,$$

et telle que l'on ait les convergences

$$\|u^\varepsilon - u\|_{L^\infty([0;T^*], L^1(0;L))} \to 0, \qquad \textit{quand } \varepsilon \to 0,$$

et pour tout $\eta \in (0,1)$,

$$\sup_{0 \leqslant t \leqslant T^*, |x - X_j(t)| \geqslant \varepsilon^\eta} |u^\varepsilon(x,t) - u(x,t)| \to 0, \qquad \textit{quand } \varepsilon \to 0.$$

De plus, on donne un développement asymptotique en ε de la solution u^ε. En effet, pour démontrer le théorème, on commence par chercher u^ε_{app}, solution approchée de (5). Loin des chocs, la solution u de (4) est régulière, donc le problème peut être approché par l'équation non visqueuse, hyperbolique. On cherche dans ces régions u^ε_{app} sous la forme

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + o(\varepsilon^2),$$

où u est la solution de (5), et u_i sont solutions d'équations hyperboliques linéaires, que l'on résout en étudiant les caractéristiques. Dans un voisinage des chocs, le terme de diffusion lisse la solution. On cherche alors u^ε sous la forme

$$u^\varepsilon(x,t) = V^j(\xi^j(x,t,\varepsilon),t) + \varepsilon V_1^j(\xi^j(x,t,\varepsilon),t) + \varepsilon^2 V_2^j(\xi^j(x,t,\varepsilon),t) + o(\varepsilon^2)$$

où l'on utilise le changement de variable $\xi^j = \frac{x-X_j(t)}{\varepsilon} + \delta^j(t)$. Les V_i^j sont solutions d'équations différentielles ordinaires du second ordre. Pour assurer de la régularité sur u^ε_{app}, on rajoute des conditions reliant les valeurs au bord des u_i et des V_i^j. u^ε_{app} s'écrit ensuite comme combinaison convexe des fonctions ainsi définies à laquelle s'ajoute un terme d'erreur d^ε. De plus, u^ε_{app} vérifie l'équation

$$\partial_t(u^\varepsilon_{app}) + \partial_x f(u^\varepsilon_{app}) - \varepsilon\partial_x^2(u^\varepsilon_{app}) - \kappa u^\varepsilon_{app} = -(R^\varepsilon(x,t))_x$$

où R^ε et ses dérivées sont bornés en $\mathcal{O}(\varepsilon^\alpha)$, $\alpha > 0$.

u^ε_{app} construit, on cherche u^ε comme étant une perturbation de u^ε_{app} pour l'équation (5). Après avoir linéarisé cette équation, on s'intéresse donc à la fonction de Green associée au linéarisé, pour laquelle on cherche des estimations. Pour cela on utilise une méthode développée par E. Grenier et F. Rousset [34], qui consiste à construire la fonction de Green de manière itérative. Là encore, on distingue les comportements au voisinage des chocs (où le caractère parabolique de l'équation domine), des comportements en dehors des chocs (où l'on suit les caractéristiques).

L'estimation ainsi obtenue sur les fonctions de Green nous permet finalement de montrer la convergence de $u^\varepsilon - u^\varepsilon_{app}$ vers 0, et donc de u^ε vers u.

La deuxième partie de cette thèse est consacrée à la mise en place d'un schéma numérique pour un système modélisant le chimiotactisme, phénomène biologique par lequel des cellules s'attirent entre elles *via* la sécrétion d'une particule chimique. En notant n la densité des cellules et c la densité de l'élément chimique, C. S. Patlak [59], et E. Keller et L. Segel [41] modélisent le chimiotactisme par les équations dites de Patlak-Keller-Segel qui, une fois adimensionnées, s'écrivent

$$\begin{cases} \partial_t n = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega, t\in\mathbb{R}^+, \quad (7)\\ \varepsilon\partial_t c = \Delta c + n - \alpha c, & x\in\Omega, t\in\mathbb{R}^+, \quad (8)\end{cases}$$

dans lesquelles $\varepsilon, \alpha \geqslant 0$ sont des constantes et Ω est un domaine de $\mathbb{R}^d$. On complète ces équations par les données au bord et initiales :

$$\nabla n\cdot\nu = 0, \quad \nabla c\cdot\nu = 0, \qquad x\in\partial\Omega, t\in\mathbb{R}^+, \tag{9}$$

$$n(t=0) \equiv n_0, \qquad x\in\Omega, \tag{10}$$

$$c(t=0) \equiv c_0 \text{ si } \varepsilon\neq 0, \qquad x\in\Omega. \tag{11}$$

On remarque ici que l'on ne met une donnée initiale sur c que quand $\varepsilon \neq 0$, c'est-à-dire, quand l'équation (8) est vraiment une équation d'évolution. Ce système a déjà fait l'objet d'études tant au niveau théorique que numérique. Ceci étant, quand $\varepsilon = 0$, le système se ramène à une équation parabolique avec un terme non local, ce qui en facilite l'étude, notamment si en plus $\alpha = 0$. L'étude théorique permet dans ce cas de mettre en évidence un phénomène d'agrégation des cellules selon une condition sur la masse totale de celles-ci. En particulier, en dimension $d = 2$, il a été montré que si la densité initiale de cellules $M := \int_\Omega n_0(x)\,\mathrm{d}x$ est inférieure à une masse critique M_0, alors la solution de (7)–(10) existe pour tout temps $t > 0$. Et que, pour des masses surcritiques $M > M_0$, la solution explose en temps fini : la densité de cellules se concentre en une masse de Dirac [8, 10]. Pour plus de précisions sur ces résultats on peut se reporter à l'article de revue de Horstmann [39] et aux références qui y sont données.

En ce qui concerne l'étude du système parabolique-parabolique (quand $\varepsilon > 0$), on trouve également une masse critique en-dessous de laquelle la solution de (7)–(11) existe pour tout temps $t > 0$ (voir [50, 29] dans le cas d'un domaine borné et [15] dans le cas de $\mathbb{R}^2$). Ceci étant, on ne sait pas dire pour l'instant avec précision ce qu'il se passe pour des masses surcritiques ($M_0 = 8\pi$ pour $\Omega = \mathbb{R}^2$, $M_0 = 4\pi$ pour Ω un domaine lisse de $\mathbb{R}^2$). Une première réponse pour $\Omega = \mathbb{R}^2$ est qu'il existe des solutions auto-similaires pour toute masse totale [9]. Une autre piste pour avancer dans le cas surcritique est de regarder ce qu'il se passe numériquement. Des travaux dans ce sens ont été effectués par des méthodes de Galerkin discontinues [25] et par des méthodes de volume fini [16]. Dans le Chapitre 3, on développe un autre schéma de type volume fini pour un système dérivé du système (7)–(11), on montre sa convergence et on donne des simulations numériques tant pour des masses sous-critiques que surcritiques.

Une des raisons qui motivent ici l'utilisation d'un schéma de type volume fini est que la première équation (7) est une équation de conservation, donc la masse de cellules $M = \int_\Omega n(x)\,\mathrm{d}x$ est conservée, propriété que l'on veut garder sur le schéma numérique. De même, on cherche à ce que le schéma conserve la positivité de n. Ceci étant, la deuxième équation n'est pas une loi de conservation. De plus, dans la première équation, ce n'est pas c qui intervient directement mais son gradient ∇c, dans un terme de transport sur n, ce qui amène d'autres difficultés. Pour les pallier, on utilise la même méthode que dans [25, 16], on dérive l'équation (8), ce qui nous donne une équation sur $S := \nabla c$. Le système devient

$$\begin{cases} \partial_t n = \Delta n - \nabla \cdot (nS), & x \in \Omega, t \in \mathbb{R}^+, \quad (12) \\ \varepsilon \partial_t S = \Delta S + \nabla n - \alpha S, & x \in \Omega, t \in \mathbb{R}^+, \quad (13) \\ \nabla \times S = 0, & x \in \Omega, t \in \mathbb{R}^+, \quad (14) \end{cases}$$

avec les données au bord et initiales

$$\nabla n \cdot \nu = 0, \quad S \cdot \nu = 0, \qquad x \in \partial\Omega, t \in \mathbb{R}^+, \tag{15}$$

$$n(t=0) \equiv n_0, \quad S(t=0) \equiv S_0, \qquad x \in \Omega. \tag{16}$$

Là encore, la deuxième équation n'est pas une équation de conservation, (sauf si $\alpha = 0$), mais cette transformation nous donne directement ce dont nous avons besoin pour la première équation : S. La méthode ensuite utilisée par A. Kurganov et A. Chertock [16] d'une part et A. Kurganov et Y. Epshteyn [25] d'autre part consiste à voir le système (12)–(13) comme un système de convection-diffusion avec terme source. Suivant l'hyperbolicité du terme convectif, ils discrétisent différemment l'équation. Plus précisément, en deux variables, ils écrivent le système sous la forme

$$\partial_t U + \partial_x F(U) + \partial_y G(U) = \Delta U + R(U)$$

et considèrent l'hyperbolicité engendrée séparément par les flux F et G. La partie convective n'étant, en réalité, jamais globalement hyperbolique, nous avons choisi de discrétiser la première équation comme une équation de transport et la seconde par un schéma centré. En l'absence des termes de diffusion, un tel schéma serait instable, mais ceux-ci permettent d'équilibrer le système. Dans le cas de conditions au bord périodiques, nous montrons par un calcul d'énergie discrète que le schéma est stable et converge vers une solution faible de (12)-(13)-(16).

Enfin, en annexe, a été mis un travail fait lors d'un stage sous la direction de Pierre Degond en 2004. Il porte sur la modélisation du trafic routier par des modèles de type dynamique des fluides. Les premiers modèles de ce type datent des années 1950 avec M. J. Lighthill et G. B. Whitham [47] et P. I. Richards [62]. Depuis, d'autres modèles, plus complexes, ont été développés. Notamment, celui de A. Aw et M. Rascle [3] a ensuite conduit au modèle présenté ici, qui est un modèle du second ordre avec contrainte.

Plus précisément, on modélise une route, sur laquelle le dépassement est impossible. On étudie l'évolution dans le temps de deux données mesurables : la densité de véhicules n ainsi que la vitesse moyenne u. On construit ensuite un modèle en considérant, d'une part, que la densité de véhicules ne peut dépasser une certaine densité maximale n^* et, d'autre part, que cette densité maximale dépend de la vitesse à laquelle on roule ($n^*(u)$). La nouveauté de ce modèle est justement ce dernier point, qui revient à prendre en compte le fait que les distances de sécurité à respecter dépendent de la vitesse. Par ailleurs, on peut observer que les conducteurs ne diminuent significativement leur vitesse que lorsque la densité de véhicules atteint la densité maximale. En prenant en compte tous ces points, on obtient le système avec contraintes :

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ (\partial_t + u\partial_x)(u + \bar{p}) = 0, \\ 0 \leqslant n \leqslant n^*(u), \bar{p} \geqslant 0, (n^*(u) - n)\bar{p} = 0. \end{cases} \tag{17}$$

Ce système peut être vu comme la limite quand $\varepsilon \to 0$ du système

$$\begin{cases} \partial_t n^\varepsilon + \partial_x(n^\varepsilon u^\varepsilon) = 0, \\ (\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(n^\varepsilon, u^\varepsilon)) = 0, \\ p(n, u) = \left(\frac{1}{n} - \frac{1}{n^*(u)}\right)^{-\gamma} \end{cases} \tag{18}$$

où $n^*(u)$ vérifie un certain nombre de propriétés.

Dans ce modèle, on voit apparaître deux types de comportement, déterminés par la présence ou l'absence de bouchons. En effet, si $n \neq n^*(u)$, les voitures avancent à leur vitesse préférée ($\bar{p} = 0$). Par contre, si on est dans un bouchon ($n = n^*(u)$), $\bar{p}$ peut être non nul et n'est *a priori* pas précisément déterminé. Une analyse assez fine des problèmes de Riemann pour le système (18), où p est remplacé par εp et, en passant à la limite, pour (17) permet d'évaluer $\bar{p}$. De plus, en approchant faiblement les données initiales par des données constantes par morceaux, on montre l'existence de solutions faibles.

Première partie

Autour de la stabilité des ondes périodiques

Certains phénomènes physiques ou biologiques sont périodiques : les ondes éléctromagnétiques, les vagues, certains mouvements cellulaires, etc. Ces structures périodiques se retrouvent dans la modélisation de ces phénomènes : soit dans les équations elles-mêmes, soit dans certaines solutions de ces équations. Dans cette partie, on s'intéresse à des problèmes de stabilité autour de ces solutions périodiques. Tout d'abord, pour des équations de conservations scalaires et périodiques, on montre que les solutions stationnaires périodiques sont asymptotiquement stables [46]. Ensuite, on montre que, dans un cadre périodique, les solutions d'un système hyperbolique sont bien limites de solutions du même problème avec viscosité évanescente.

Ces deux chapitres font l'objet de publications :

LE BLANC, V. L^1-stability of periodic stationary solutions of scalar convection-diffusion equations. *J. Differential Equations 247*, 6 (2009), 1746–1761.

LE BLANC, V. Persistence of generalized roll-waves under viscous perturbation. Soumis à *SIAM J. Math. Anal.*, 2010.

Chapitre 1

L^1-stability of periodic stationary solutions of scalar convection-diffusion equations

Abstract. The aim of this paper is to study the L^1-stability of periodic stationary solutions of scalar convection-diffusion equations. We obtain dispersion in L^2 for all space dimensions using Kružkov type entropy. And when the space dimension is one, we estimate the number of sign changes of a solution to obtain L^1-stability.

1.1 Introduction

We study the solutions of a scalar convection-diffusion equation of the form:

$$\partial_t u + \operatorname{div}(f(u,x)) = \Delta u, \quad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

where $x \mapsto f(\cdot, x)$ is an Y-periodic function with $Y = \prod_{i=1}^d (0, T_i)$ the basis of a lattice. We assume that f belongs to $\mathcal{C}^2(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$. For this equation, periodic stationary solutions w_p exist and are parameterized by their space average p: this is a result of A.-L. Dalibard in [19]. In this paper, we focus on the L^1-stability of these periodic stationary solutions.

When f only depends on u, the periodic stationary solutions are the constants and the L^1-stability of the constants is already proved by H. Freistühler and D. Serre in the one-dimensional space case in [28] and by D. Serre in all space dimensions in [65]. We define the space

$$L^1_0(\mathbb{R}^d) = \left\{ u \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 0 \right\}.$$

With this notation, the result can be written as follows:

Theorem 1.1. (See [65].) *For all $k \in \mathbb{R}, b \in L^1_0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the unique solution $u \in L^\infty_{loc}(\mathbb{R}^+, L^\infty(\mathbb{R}^d))$ of*

$$\begin{cases} \partial_t u + \mathrm{div}(f(u)) = \Delta u, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = k + b(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

satisfies

$$\lim_{t\to\infty} \|u(t,\cdot) - k\|_1 = 0.$$

The proof of this result can be made in 3 steps. First, the global existence of solution of (1.2) is proved using the Duhamel's formula with $\mathrm{div}(f(u))$ as a perturbation of the heat equation, one obtains

$$u(t) = K^t * u_0 + \int_0^t \nabla K^{t-s} * f(u(s))\,\mathrm{d}s.$$

The maximum principle allows to conclude about global existence by induction. This defines the nonlinear semigroup $\tilde{S}^t$ so that $u(t) = \tilde{S}^t u_0$ is the solution of (1.2).

Secondly, one establishes the so-called four "Co-properties" for u_0, v_0 in $L^\infty(\mathbb{R}^d)$:

1. Comparison: $u_0 \leqslant v_0$ a.e.$\Rightarrow \tilde{S}^t u_0 \leqslant \tilde{S}^t v_0$ a.e.;
2. Contraction: $v_0 - u_0 \in L^1(\mathbb{R}^d) \Rightarrow \tilde{S}^t v_0 - \tilde{S}^t u_0 \in L^1(\mathbb{R}^d)$ and
$$\|\tilde{S}^t v_0 - \tilde{S}^t u_0\| \leqslant \|v_0 - u_0\|;$$
3. Conservation (of mass): $v_0 - u_0 \in L^1(\mathbb{R}^d) \Rightarrow \tilde{S}^t v_0 - \tilde{S}^t u_0 \in L^1(\mathbb{R}^d)$ and
$$\int_{\mathbb{R}^d} (\tilde{S}^t v_0 - \tilde{S}^t u_0) = \int_{\mathbb{R}^d} (v_0 - u_0);$$
4. Constants: if u_0 is a constant, then $\tilde{S}^t u_0 \equiv u_0$.

Two methods allow to conclude: one in one space dimension and another one in all space dimensions. The first one is due to H. Freistühler and D. Serre [28]: they study the number of sign changes of the solution. Having assumed that $k = 0, f(0) = 0, f'(0) = 0$ they study the primitive V of the solution u which vanishes at

$-\infty$: $V(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y$. Since $b \in L^1_0(\mathbb{R})$, this primitive also vanishes at $+\infty$ and belongs to $L^\infty(\mathbb{R})$. Moreover, V satisfies a parabolic equation

$$\partial_t V + f(\partial_x V) = \partial_x^2 V.$$

They also apply the lemma of H. Matano [49] on V to estimate the number of sign changes of the derivative of V: u. Estimates on both $\|u(t)\|_{L^1}$ by $\|V(t)\|_{L^\infty}$ follows. Using L^2-estimates on the equations on both u and V, one shows that

$$\lim_{t\to\infty} \|V(t)\|_{L^\infty} = 0,$$

which permits to obtain the theorem.

The second method, due to D. Serre [65], is based on the Duhamel's formula. A dispersion inequality is obtained using the entropy $u \mapsto u^2$ for equation (1.2) and L^1-contraction, one obtains

$$\|\tilde{S}^t u_0\|_2 \leqslant c_d \frac{\|u_0\|_1}{t^{d/4}}.$$

Under the rather general assumption that $f(u)$ is bounded by $|u|^2$,we prove

$$\lim_{t\to\infty} \|\tilde{S}^t b\|_1 = 0$$

combining dispersion estimate and estimates on the heat kernel.

In this article, we will see how we can adapt some of these arguments to the case where f depends both on u and x. We recall that in this case the stationary solutions w_p considered are periodic, parameterized by their space average p.

We obtain one theorem in the one-dimensional space case:

Theorem 1.2. *For all $p \in \mathbb{R}$, for all $b \in L^1_0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the unique solution u in $L^\infty_{loc}(\mathbb{R}^+, L^\infty(\mathbb{R}))$ of*

$$\begin{cases} \partial_t u + \operatorname{div}(f(u,x)) = \Delta u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = w_p + b(x), & x \in \mathbb{R}, \end{cases}$$

satisfies:

$$\lim_{t\to\infty} \|u(t,\cdot) - w_p\|_1 = 0.$$

First, we observe that in this theorem we assume $\int_{-\infty}^{\infty} b(x)\,\mathrm{d}x = 0$. This assumption is necessary because of the conservation of mass:

$$\int_{\mathbb{R}^d} (v - w_p) = \int_{\mathbb{R}^d} (v_0 - w_p) = \int_{\mathbb{R}^d} b.$$

Actually, we cannot have L^1-convergence when $\int_{\mathbb{R}^d} b \neq 0$. But this assumption is not necessary to prove L^p-convergence for $1 < p \leqslant 2$ and in this case we obtain a rate of convergence $d/2(1-1/p)$.

To prove the theorem, we use results on the nonlinear semigroup and the lemma of H. Matano, as in [28]. The main difference with the proof of D. Serre and H. Freistühler ([65, 28]) appears in the proof of L^2-estimates for u and its primitive V. Since the problem is inhomogeneous, $u \mapsto u^2$ is not an entropy and we have to find a new entropy to prove dispersion inequality. For V the results on periodic stationary solutions of A.-L. Dalibard permit to prove that $\|V\|_2$ is bounded.

The paper is organized as follows. In Section 2, we recall the result obtained by A.-L. Dalibard in [19] about the existence of periodic stationary solutions. In Section 3, we focus on the existence and the properties of our nonlinear semigroup in all space dimensions: comparison principle, contraction in L^1, conservation of mass, dispersion inequality. For its existence and its three first properties the proofs are similar to the homogeneous case $f(u,x) = f(u)$, except that the maximum principle does not hold anymore and is replaced by a comparison principle. For the dispersion inequality, we build a new type of Kružkov entropy, based on periodic stationary solutions instead of constants. In Section 4, we focus on the one-dimensional space case, and prove Theorem 1.2 using the lemma of H. Matano about the number of sign changes.

1.2 Existence of stationary solutions

In this section, we recall the existence result of A.-L. Dalibard [19]. When f depends only on u, but not on x, i.e. when we are in the case studied by D. Serre in [65], the stationary solutions considered are all the constants. But in our case the constants are not solutions except if $\operatorname{div}(f(k,x)) = 0$ for all $x \in \mathbb{R}^d$. The existence of another class of stationary solutions is proved by A.-L. Dalibard (see Theorem 2 and Lemma 6 in [19]): there exist periodic stationary solutions, indexed by their space average.

In this section, we recall a part of her results for the following equation:

$$\operatorname{div}(f(u,x)) = \Delta u, x \in \mathbb{R}^d$$

where $x \mapsto f(\cdot, x)$ is an Y-periodic function with $Y = \prod_{i=1}^d (0, T_i)$ the basis of a lattice. We note the space average of a function u: $\langle u \rangle_Y = \frac{1}{|Y|} \int_Y u(x)\,\mathrm{d}x$.

Theorem 1.3. *Let $f = f(u,x) \in \mathcal{C}^2(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$ such that $\partial_u f \in L^\infty(\mathbb{R} \times Y)$. Assume that there exist $C_0 > 0$, and $n \in [0, \frac{d+2}{d-2})$ when $d \geqslant 3$, such that for all $(p,x) \in \mathbb{R} \times Y$*

$$|\operatorname{div} f(p,x)| \leqslant C_0(1 + |p|^n).$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $w(\cdot,p) \in H^1_{per}(Y)$ of

$$-\Delta w(x,p) + \operatorname{div} f(w(x,p),x) = 0, \text{ such that } \langle w(\cdot,p)\rangle_Y = p.$$

For all $p \in \mathbb{R}, w(\cdot,p)$ belongs to $W^{2,q}_{per}(Y)$ for all $1 < q < \infty$ and for all $R > 0$, there exists $C_R > 0$ such that

$$\|w(\cdot,p)\|_{W^{2,q}(Y)} \leqslant C_R \quad \forall p \in \mathbb{R}, |p| \leqslant R,$$

$C_R > 0$ depending only on d, Y, C_0, n, q, p_0 and R.

Furthermore, for all $p \in \mathbb{R}, \partial_p w(\cdot,p) \in H^1_{per}(Y)$ is in the kernel of the linear operator

$$-\Delta + \operatorname{div}(\partial_u f(w(x,p),x)\cdot) = 0, \text{ and } \langle \partial_p w\rangle_Y = 1.$$

And there exists $\alpha > 0$ depending only on d, Y and $\|\partial_u f\|_\infty$ such that

$$\partial_p w(x,p) > \alpha \text{ for a.e. } (x,p) \in Y \times \mathbb{R}.$$

Hence,

$$\lim_{p\to+\infty} \inf_Y w(x,p) = +\infty,$$
$$\lim_{p\to-\infty} \sup_Y w(x,p) = -\infty.$$

Remarks 1.1.

- A consequence of this theorem is that for all $x \in \mathbb{R}^d$, the application $p \mapsto w(p,x)$ is increasing and bijective from $\mathbb{R}$ to $\mathbb{R}$.
- In this theorem, we impose the restrictive assumption that $\partial_u f \in L^\infty$ on the whole domain $\mathbb{R} \times Y$. When $\partial_u f$ belongs only to $L^\infty_{loc}(L^\infty(Y))$, we obtain that $\partial_p w > 0$ but we have not the existence of the constant α. Hence, we have no result on the limit when $p \to \pm\infty$ of $\inf_Y w(x,p)$ and $\sup_Y w(x,p)$, but we have that the application

  $$\begin{array}{rcl} \mathbb{R} & \to & \left]\lim_{p\to+\infty} \inf_Y w(x,p), \lim_{p\to-\infty} \sup_Y w(x,p)\right[\\ p & \mapsto & w(p,x) \end{array}$$

  is bijective. And we can adapt the result of Theorem 1.2 in this case: we just have to make the assumption that there exists p such that for all $x \in \mathbb{R}^d, u_0(x) \in [w(-p,x), w(p,x)]$.

In the sequel, we use the notation: $w_p = w(\cdot,p)$.

1.3 The nonlinear semigroup

In what follows, we focus on the Cauchy problem for equation (1.1):

$$\begin{cases} \partial_t u + \operatorname{div}(f(u,x)) = \Delta u, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

where the initial datum u_0 belongs to $L^\infty(\mathbb{R}^d)$. First, we adapt the approach of D. Serre [65] to prove the existence of solutions and their properties: comparison principle, L^1-contraction, conservation of mass. Then, we prove a dispersion inequality, using a new type of entropy based on periodic solutions.

1.3.1 Existence of the nonlinear semigroup

As in [65], the proof of the existence of solutions is based on Duhamel's formula for heat equation. We also need a comparison principle to replace the maximum principle which is not true here.

Let us write problem (1.3) in the form:

$$\begin{cases} \partial_t u - \Delta u = -\operatorname{div}(f(u,x)), & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.4}$$

Here, the heat operator appears in the left-hand side of (1.4), and the right-hand side is a lower order perturbation. Denote by H^t the heat semigroup and K^t its kernel. They are given by:

$$H^t u_0 = K^t * u_0, \quad K^t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x\|^2}{4t}\right)$$

and satisfy the following properties:

$$\|H^t u_0\|_p \leqslant \|u_0\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{1.5}$$

$$\|\nabla_x H^t u_0\|_p \leqslant c'_p t^{-\frac{1}{2}} \|u_0\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{1.6}$$

$$\int_{\mathbb{R}^d} K^t(x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}^d} \nabla_x K^t(x)\,\mathrm{d}x = 0. \tag{1.7}$$

We prove the following result:

Proposition 1.1. *Assume that $f \in \mathcal{C}^k(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$. Then for all $a \in L^\infty(\mathbb{R}^d)$, there exist $T > 0$ and a unique solution $u \in L^\infty([0,T] \times \mathbb{R}^d)$ of (1.3). Moreover, $u \in \mathcal{C}^k((0,T), \mathcal{C}^\infty(\mathbb{R}^d))$ and T depends only on $\|u_0\|_\infty$.*

Proof. We are searching for the mild solution of (1.3),i.e. which verifies Duhamel's formula:

$$\begin{aligned} u(t,\cdot) &= K^t * u_0 - \int_0^t K^{t-s} * \mathrm{div}(f(u(s,\cdot),\cdot))\,\mathrm{d}s \\ &= K^t * u_0 - \int_0^t \nabla_x K^{t-s} * f(u(s,\cdot),\cdot)\,\mathrm{d}s. \end{aligned}$$

Hence, we search for the solution of (1.3) as a fixed point of the map

$$M : u \mapsto \Big(t \mapsto K^t * u_0 - \int_0^t \nabla_x K^{t-s} * f(u(s,\cdot),\cdot)\,\mathrm{d}s \Big).$$

In order to use Picard's fixed point theorem we need to find a space which is stable by M and where M is contractant. Using (1.5)-(1.6) with $p = \infty$ we have the following estimate for all $u \in L^\infty(\mathbb{R}^d)$:

$$\|Mu(t)\|_\infty \leqslant \|u_0\|_\infty + \int_0^t \frac{c'_\infty}{(t-s)^{1/2}} \|f(u(s,\cdot),\cdot)\|_\infty\,\mathrm{d}s.$$

We assume that for all $0 \leqslant\ s \leqslant\ T, \|u(s)\|_\infty \leqslant 2\|u_0\|_\infty$. Since $f(\cdot,x)$ is locally in L^∞, uniformly in x, there exists C such that for all $0 \leqslant s \leqslant T$,

$$\|f(u(s,\cdot),\cdot)\|_\infty \leqslant C,$$

where C does not depend on u, but only on $\|u\|_{L^\infty((0,t)\times\mathbb{R}^d)} \leqslant 2\|u_0\|_\infty$. Therefore, we obtain the following estimate

$$\|Mu(t)\|_\infty \leqslant \|u_0\|_\infty + 2c'_\infty C\sqrt{T}, \quad \forall 0 \leqslant t \leqslant T.$$

For T sufficiently small ($2c'_\infty C\sqrt{T} < \|u_0\|_\infty$), the map M preserves the ball of radius $2\|u_0\|_\infty$ of $L^\infty((0,T)\times\mathbb{R}^d)$. This ball is denoted $B(2\|u_0\|_\infty)$. Next we prove that M is a contraction: let $u, v \in B(2\|u_0\|_\infty)$, then

$$Mv(t) - Mu(t) = \int_0^t \nabla_x K^{t-s} * (f(u(s,\cdot),\cdot) - f(v(s,\cdot),\cdot))\,\mathrm{d}s.$$

Since $f(\cdot,x)$ is locally Lipschitz, uniformly in x, there exists C' (depending on $2\|u_0\|_\infty$) such that $\|f(u,\cdot) - f(v,\cdot)\|_\infty \leqslant C'\|u-v\|_\infty$. Hence, we obtain

$$\|Mu - Mv\|_\infty \leqslant 2c'_\infty C'\sqrt{T}\|u-v\|_\infty$$

and for T small enough, the map M is stable and contractant on $B(2\|u_0\|_\infty)$.
We can now use Picard's fixed point theorem to obtain a unique local solution in $L^\infty([0,T]\times\mathbb{R}^d)$. Moreover, using again Duhamel's formula, we prove that this solution is regular in time if f is regular in u and x ; for instance u is in $\mathcal{C}^k((0,T),\mathcal{C}^\infty(\mathbb{R}^d))$ if f is in $C^k(\mathbb{R},\mathcal{C}^1(\mathbb{R}^d))$. □

To prove global existence in homogeneous problem, one uses maximum principle. When the problem is inhomogeneous, this maximum principle is false and one uses a comparison principle:

Lemma 1.2. Comparison principle: *Let $u, v \in L^\infty([0,T] \times \mathbb{R}^d)$ two solutions of (1.1) on $(0,T)$ such that for all $x \in \mathbb{R}^d, u_0(x) \leqslant v_0(x)$. Then for all $t \in [0,T]$, and $x \in \mathbb{R}^d$, we have $u(t,x) \leqslant v(t,x)$.*

Using this lemma, we then prove global existence of solution:

Proposition 1.3. *Assume that $f \in \mathcal{C}^k(\mathbb{R}, \mathcal{C}^1(\mathbb{R}^d))$. Then for all $u_0 \in L^\infty(\mathbb{R}^d)$, there exists a unique solution $u \in \mathcal{C}^k(\mathbb{R}^+, \mathcal{C}^\infty(\mathbb{R}^d))$ of (1.3).*

Proof. From Theorem 1.3 and Remarks 1.1 we deduce that for all x, the application $p \mapsto w_p(x)$ is invertible from $\mathbb{R}$ to $\mathbb{R}$. Since $u_0 \in L^\infty(\mathbb{R}^d)$, there exists p such that $w_{-p}(x) \leqslant u_0(x) \leqslant w_p(x)$. Proposition 1.1 gives us T (we can chose $T = T(\max\{\|w_{-p}\|_\infty, \|w_p\|_\infty\}))$ and a unique solution u. The lemma implies that for all $t \in (0,T)$, and $x \in \mathbb{R}$, we have $w_{-p}(x) \leqslant u(t,x) \leqslant w_p(x)$. Therefore, we can iterate the local existence to prove that u exists on $(0,T), \ldots, (kT, (k+1)T)$ for any $k \in \mathbb{N}$. Finally, we obtain a unique bounded solution, global and smooth for positive time. □

Next, we define the nonlinear semigroup S^t on $L^\infty(\mathbb{R}^d)$. From now, we will note $u = S^t u_0, v = S^t v_0$ if $u_0, v_0 \in L^\infty(\mathbb{R}^d)$.

As in [65], we have some properties on this semigroup: we have already mentionned the comparison principle (Lemma 1.2). We also have L^1-contraction and conservation of mass. And as said above, the constants are no longer stationary solutions: they are replaced by periodic functions.

Proposition 1.4. *For all $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ such that $u_0 - v_0 \in L^1(\mathbb{R}^d)$, for all $t > 0$ we have*

(i) L^1-contraction: $S^t u_0 - S^t v_0 \in L^1(\mathbb{R}^d)$ *and* $\|S^t u_0 - S^t v_0\|_1 \leqslant \|u_0 - v_0\|_1$;

(ii) conservation of mass: $\displaystyle\int_{\mathbb{R}^d} (S^t u_0 - S^t v_0) = \int_{\mathbb{R}^d} (u_0 - v_0)$.

Proof. Let $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ such that $u_0 - v_0 \in L^1(\mathbb{R}^d)$. We first prove that $S^t u_0 - S^t v_0 \in L^1(\mathbb{R}^d)$. Using Duhamel's formula, one obtains:

$$v(t) - u(t) = K^t * (v_0 - u_0) - \int_0^t (\nabla_x K^{t-s}) * (f(v(s,\cdot),\cdot) - f(u(s,\cdot),\cdot))\, \mathrm{d}s. \quad (1.8)$$

Taking the L^1-norm and using estimates (1.5)-(1.6) for $p = 1$, we deduce that

$$\sup_{s\leqslant t} \|v(s) - u(s)\|_1 \leqslant \|v_0 - u_0\|_1 + 2c_1' C' \sqrt{t} \sup_{s\leqslant t} \|v(s) - u(s)\|_1.$$

Hence, for t small enough, $v(s) - u(s) \in L^1(\mathbb{R}^d)$, for all $0 \leqslant s \leqslant t$ and by induction it is true for all $t \in \mathbb{R}^+$.
We now prove the L^1-contraction principle. For all $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ one shows that

$$\partial_t |u - v| + \operatorname{div}(\operatorname{sgn}(u - v)(f(u, \cdot) - f(v, \cdot))) \leqslant \Delta |u - v|.$$

Noting

$$w = -K^t * |v_0 - u_0| + \int_0^t \partial_x K^{t-s} * \operatorname{div}((f(u, x) - f(v, x)) \operatorname{sgn}(u - v)) + |u - v|, \quad (1.9)$$

we easily prove $\partial_t w \leqslant \Delta w$ and $w(0) = 0$. Using comparison principle, we have $w \leqslant 0$. We integrate (1.9) according to x to obtain

$$0 \geqslant \int_{\mathbb{R}^d} w = -\int_{\mathbb{R}^d} |v_0 - u_0| + \int_{\mathbb{R}^d} |u - v|. \quad (1.10)$$

From (1.10), we deduce the contraction principle.
Let us now prove the conservation of mass. Integrating (1.8), and using (1.7) we immediately obtain for all $u_0, v_0 \in L^\infty(\mathbb{R}^d)$: $\partial_t \int_{\mathbb{R}^d} (u - v) = 0$ and

$$\int_{\mathbb{R}^d} (u - v) = \int_{\mathbb{R}^d} (u_0 - v_0).$$

□

1.3.2 Dispersion inequality

In this section, we prove the following dispersion inequality for equation (1.1):

Proposition 1.5. *Let $R \in \mathbb{R}$. There exists $C > 0$ so that for all $p \in \mathbb{R}, b \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that $w_{-R} \leqslant w_p + b \leqslant w_R$, $u(t) = S^t(w_p + b)$ verifies a dispersion inequality:*

$$\|u(t) - w_p\|_2 \leqslant C_d \frac{\|b\|_1}{t^{d/4}}. \quad (1.11)$$

This estimate gives convergence in L^2 when $u_0 - w_0 \in L^1(\mathbb{R}^d)$ and the speed of this convergence. In Section 1.4, we will see how L^2-convergence imply L^1-convergence in the one dimensional space case.

This property is first proved by P. Bénilan and C. Abourjaily in [1] in the case where f does not depend on x. When $\tilde{S}^t$ denotes the semigroup of (1.2), their result can be written as follows:

$$\|\tilde{S}^t u_0\|_2 \leqslant c_d \frac{\|u_0\|_1}{t^{d/4}}.$$

In this case, the proof of the inequality is based on the fact that for all convex function η, there exists g such that for all u, $\eta'(u)\operatorname{div}(f(u)) = \operatorname{div}(g(u))$, in particular for $\eta(u) = u^2$. This property is false in our case but we still have a dispersion inequality (1.11).

To prove Proposition 1.5, we use a new class of entropies. When f does not depend on x, an interesting class of entropies is the Kružkov entropies $u \mapsto |u - k|$ with $k \in \mathbb{R}$. Those are convex functions and for all solutions u of (1.2), we have the inequality

$$\partial_t |u - k| + \operatorname{div}(\operatorname{sgn}(u - k)(f(u) - f(k))) \leqslant \Delta |u - k|.$$

This inequality is still true in our case but we do not want to compare our solutions to constants anymore, because they are not stationary solutions of (1.3). Hence, we define a new type of entropy, using the stationary solutions w_p.

Proof. Without loss of generality we assume that $p = 0$. We have just said that we need to base our new entropy on the stationary solutions. Theorem 1.3 gives us that for all $p \in \mathbb{R}$, there exists a unique stationary solution w_p under the constraint $\langle w_p \rangle_Y = p$. Following the construction of Kružkov entropies, let us consider , for any $p \in \mathbb{R}$, the function η_p such that

$$\eta_p : (x, u) \mapsto \eta_p(x, u) = |u - w_p(x)|.$$

This application verifies the inequality:

$$\partial_t \eta_p(u(t, x), x) + \operatorname{div}(\operatorname{sgn}(u - w_p)(f(u, x) - f(w_p, x))) \leqslant \Delta \eta_p.$$

In order to define our new entropy η, we define two auxiliary functions $p(u, x)$ and $\pi(x, t)$. We recall that for all $x \in \mathbb{R}^d$, the function $p \mapsto w_p(x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. We note $p(u, x)$ the inverse of this application. It verifies:

$$\forall x \in \mathbb{R}^d, u \in \mathbb{R}, w_{p(u,x)}(x) = u.$$

If u is a function defined on $\mathbb{R}^+ \times \mathbb{R}^d$, we define $\pi(t, x) = p(u(t, x), x)$. One remarks that $-R \leqslant \pi \leqslant R$. We can now define our particular entropy η as:

$$\eta(u, x) = \int_0^{p(u,x)} (u - w_p(x))\, \mathrm{d}p.$$

This function is non negative. Next, we derive energy estimate on u using this new entropy. Deriving $\eta(u(t,x),x)$ with respect to t and using (1.3), one obtains

$$\partial_t(\eta(u(t,x),x)) = \int_0^{\pi(t,x)} \Delta(u - w_p)\,\mathrm{d}p - \int_0^{\pi(t,x)} \operatorname{div}(f(u,x) - f(w_p,x))\,\mathrm{d}p. \quad (1.12)$$

The last term of (1.12) is written as:

$$\int_0^{\pi(t,x)} \operatorname{div}(f(u,x) - f(w_p,x))\,\mathrm{d}p = \operatorname{div}\left(\int_0^{\pi(t,x)} (f(u,x) - f(w_p,x))\,\mathrm{d}p\right)$$

and

$$\int_0^{\pi(t,x)} \Delta(u - w_p)\,\mathrm{d}p = \Delta(\eta(u(t,x),x)) - \nabla\pi \cdot \nabla(u - w_p)|_{p=\pi(t,x)}.$$

We then obtain the following partial differential equation:

$$\partial_t\eta(u) + \operatorname{div}\left(\int_0^{\pi(t,x)} (f(u,x) - f(w_p,x))\,\mathrm{d}p\right) = \Delta\eta(u) - \nabla\pi \cdot \nabla(u - w_p)|_{p=\pi(t,x)}. \quad (1.13)$$

Moreover, we have the equality:

$$0 = \nabla(u(t,x) - w_{\pi(t,x)}(x)) = \nabla(u - w_p)|_{p=\pi(t,x)} - \partial_p w_\pi \cdot \nabla\pi. \quad (1.14)$$

We deduce from (1.13) and (1.14) that η satisfies the equation

$$\partial_t\eta(u) + \operatorname{div}\left(\int_0^{\pi(t,x)} (f(u,x) - f(w_p,x))\,\mathrm{d}p\right) = \Delta\eta - \partial_p w_\pi \cdot |\nabla\pi|^2. \quad (1.15)$$

Integrate equation (1.15) in space: we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} \partial_p w_\pi |\nabla\pi|^2 = 0.$$

Moreover, Theorem 1.3 gives us $\partial_p w_\pi \geqslant \alpha > 0$. Using this inequality and Nash inequality [67]:

$$\|\pi\|_2 \leqslant c_d \|\pi\|_1^{(1-\theta)} \|\nabla\pi\|_2^\theta \quad \text{where} \quad \frac{1}{\theta} = 1 + \frac{2}{d},$$

we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x + C_d \frac{\|\pi\|_2^{2/\theta}}{\|\pi\|_1^{2(1-\theta)/\theta}} \leqslant 0. \quad (1.16)$$

Let us now relate π with η:

$$\eta(u(t,x),x) = \int_0^{\pi(t,x)} (u(t,x) - w_p(x))\,\mathrm{d}p.$$

From the estimate

$$\begin{aligned} |u(t,x) - w_p(x)| = |w_{\pi(t,x)}(x) - w_p(x)| = \left| \int_p^{\pi(t,x)} \partial_p w_p(x)\,\mathrm{d}p \right| \\ \leqslant |\pi(t,x)| \sup_p |\partial_p w_p|, \end{aligned} \tag{1.17}$$

we deduce,

$$\eta(u(t,x),x) \leqslant |\pi(t,x)|^2 \sup_p |\partial_p w_p|.$$

Since $\partial_p w_p$ is locally bounded in p, i.e. $\partial_p w_p(x) \leqslant C$ for all $x \in \mathbb{R}^d$, for all $p \in [-R,R]$, we deduce the inequality:

$$\eta(u(t,x),x) \leqslant C|\pi(t,x)|^2. \tag{1.18}$$

We combine (1.16) and (1.18) to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x\right) + C\frac{(\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x)^{1/\theta}}{\|\pi\|_1^{2(1-\theta)/\theta}} \leqslant 0.$$

We have now to overvalue $\|\pi\|_1$ uniformly in t. Now

$$\pi(t,x) = p(u(t,x),x) - p(w_0(x),x) = \int_{w_0(x)}^{u(t,x)} \partial_u p(w,x) dw.$$

We deduce from the minoration $\partial_p w_p \geqslant \alpha$ the estimate $\partial_u p \leqslant 1/\alpha$ and we deduce

$$\|\pi(t)\|_1 \leqslant \frac{1}{\alpha}\|u(t) - w_0\|_1 \leqslant \frac{1}{\alpha}\|b\|_1$$

with L^1-contraction. Finally, we have the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x\right) + \frac{C}{\|b\|_1^{2(1-\theta)/\theta}}\left(\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x\right)^{1/\theta} \leqslant 0. \tag{1.19}$$

Using $g := -\left(\int_{\mathbb{R}^d} \eta(u)(x)\,\mathrm{d}x\right)^{1-1/\theta}$, we solve this inequality and we obtain

$$g(t) \leqslant (1 - 1/\theta)C\frac{t}{\|b\|_1^{2(1-\theta)/\theta}}.$$

Hence,

$$\left(\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x\right) \leqslant C' \frac{\|b\|_1^2}{t^{d/2}}.$$

To conclude the proof, we prove that there exists $C > 0$ such that for all $t \geqslant 0$, $\sqrt{\int \eta(u(t))(x)} \geqslant C\|u(t) - w_0\|_2$. First, we have

$$\begin{aligned}
\eta(u)(x) &= \int_0^{p(u(x),x)} (u(x) - w_p(x))\,\mathrm{d}p \\
&= \int_0^{p(u(x),x)} \left(\int_p^{p(u(x),x)} \partial_p w_q(x) dq\right) \mathrm{d}p \\
&\geqslant \alpha \int_0^{p(u(x),x)} (p(u(x),x) - p)\,\mathrm{d}p \\
&= \alpha \frac{p(u(x),x)^2}{2}.
\end{aligned}$$

Then, estimate (1.17) shows that:

$$|u - w_0|^2 \leqslant (\sup_p |\partial_p w_p|)^2 p(u(x),x)^2 \leqslant C^2 p(u(x),x)^2.$$

This concludes the proof of the theorem. □

1.4 One-dimensional space case: proof of Theorem 1.2

In this section, we prove L^1-convergence in one space dimension. We bypass the utilization of Duhamel's formula by counting the number of sign changes. This method is used by H. Freistühler and D. Serre in [28] to prove that constants are stable in L^1 when the flux f does not depend on x, and when the space dimension is one. It uses a lemma of H. Matano [49] which gives an evaluation of the number of sign changes for the solution of our equation. The proof is carried out in four steps: (1) At first, we make additional assumptions on f and the initial datum. (2) Then, we prove L^2-estimates on u and its primitive V and we deduce that $\|V(t)\|_\infty$ vanishes at $+\infty$. (3) Using lemma of H. Matano, we find that $\|u(t)\|_1$ is controlled by $\|V(t)\|_\infty$, so we prove the result under the additional hypothesis. (4) We generalized the result without these assumptions.

Proof. First, up to a translation, we will assume that

$$p = 0, w_p \equiv 0 \text{ and } f(0,\cdot) \equiv 0.$$

We define $F(u,x) = f(u,x) - \partial_u f(0,x)u$ which verifies: $F(0,\cdot) \equiv 0$, and $\partial_u F(0,\cdot) \equiv 0$ and we deduce the inequality

$$F(u,x) \leqslant \frac{|u|^2}{2} \sup |\partial_u^2 F|.$$

(1) Let us first assume that b is bounded in the following sense: let

$$p^+ = \min\{p : b \leqslant w_p\}, p^- = \max\{p : b \geqslant w_p\},$$

we assume that

$$\max\{\|w_{p^+}\|_\infty, \|w_{p^-}\|_\infty\} < r.$$

Therefore, we have: $|b| < r$ and using the comparison property for all t, $|S^t b| < r$. Moreover, we assume $\|b\|_1 \sup_{[-r,r]} |\partial_u^2 F| \leqslant 1$. We will see at the end of the proof how to remove these assumptions.
We further assume that $b \in \mathcal{C}_0^\infty(\mathbb{R}, [-r,r]), l(b) < \infty$ where $l(b)$ is the number of sign changes of b. Actually, we can approximate every function b that verifies the conditions of step (1) by a function in $\mathcal{C}_0^\infty$; and since the support is compact, we can suppose that the sign of the function changes only a finite number of time.

(2) Assume now that b verifies all the previous assumptions. We define $V(x) = \int_{-\infty}^x u(t,y)\,\mathrm{d}y$. Since u belongs to L^1, V is well defined and belongs to L^∞ and $\|V\|_\infty \leqslant \|b\|_1$. Moreover, since $\int_\mathbb{R} b = 0$ and we have mass conservation, we have that $V \in \mathcal{C}_0^\infty$. In search of estimates on V, we consider an equation verified by V:

$$\partial_t V + \partial_u f(0,x)\partial_x V + F(\partial_x V, x) = \partial_x^2 V. \tag{1.20}$$

Let $\theta : x \mapsto \theta(x)$ from $\mathbb{R}$ to $\mathbb{R}$ be a positive function (which will be specified later). Multiplying by θV and integrating in space, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\theta V^2 + \int \theta |\partial_x V|^2 = -\int \theta V F(\partial_x V, x) + \int \frac{V^2}{2}\left(\partial_x(\theta \partial_u f(0,x))\partial_x^2 \theta\right).$$

Besides, we have the inequality: $|F(\partial_x V, x)| \leqslant \frac{|\partial_x V|^2}{2} \sup |\partial_u^2 F|$. We deduce the estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \theta V^2\right) \leqslant -\int \theta |\partial_x V|^2 + \int V^2 \left(\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta\right).$$

Now we choose θ to obtain an estimate on $\int \theta V^2$. We impose:

– $\theta > \alpha > 0$ so that $V \mapsto \int \theta V^2$ is a norm on L^2.
– $\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta = 0$.
Actually, we only need that $\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta \leqslant 0$.

The following lemma ensures the existence of such a θ:

Lemma 1.6. *There exists $\theta > 0$ in $H^1_{per}(Y)$ such that*

$$\partial_x(\theta \partial_u f(0,x)) + \partial_x^2 \theta = 0.$$

Proof. We focus on the equation:

$$\partial_t w - \partial_x(f(w,x)) = \partial_x^2 w.$$

Theorem 1.3 ensures the existence of a periodic stationary solution $\tilde{w}_p$ of space average p and this one verifies: $\partial_p \tilde{w}_p > 0$. Moreover, the function defined by $\theta \equiv \partial_p \tilde{w}_p|_{p=0}$ is Y-periodic, in H^1 and verifies the following equation:

$$\partial_x(\theta \partial_v f(\tilde{w}_0, x)) + \partial_x^2 \theta = 0.$$

We remark that $\partial_x f(0,x) = 0 = \partial_x^2 0$. Since $\tilde{w}_0$ is the unique function such that $\partial_x^2 \tilde{w}_0 = -\partial_x f(\tilde{w}_0, x)$ and $\langle \tilde{w}_0 \rangle_Y = 0$, we have $\tilde{w}_0 \equiv 0$. $\square$

The definition of θ ensures the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \theta V^2\right) \leqslant -\int \theta |\partial_x V|^2.$$

Since θ belongs to $H^1_{per}(Y) \subset \mathcal{C}(\mathbb{R})$, there exists $c > 0$ such that $c < \theta$. Hence, we deduce that V is bounded in $L^2(\mathbb{R})$:

$$c\int |V|^2(t) \leqslant \int \theta |V|^2(t) \leqslant \int \theta |V|^2(0). \tag{1.21}$$

We also have an estimate on $\|u\|_2$. Indeed, we proved in Section 4 the dispersion inequality (1.11) for u:

$$\left(\int_{\mathbb{R}} |u(x,t)|^2 \,\mathrm{d}x\right) \leqslant C_1 \frac{\|b\|_1^2}{t^{1/2}}.$$

We deduce that

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{1.22}$$

We can now prove an estimate on $\|V\|_\infty$. We have

$$V^2(x,t) = 2\int_{-\infty}^{x} u(y,t)V(y,t)\,\mathrm{d}y \leqslant 2\|u(\cdot,t)\|_2 \|V(\cdot,t)\|_2.$$

From equations (1.22) and (1.21), we deduce:

$$\lim_{t\to\infty} \|u(\cdot,t)\|_2 = 0, \|V(\cdot,t)\|_2 \text{ uniformly bounded in } t.$$

Consequently $\lim_{t\to\infty} \|V(\cdot,t)\|_\infty = 0$.

(3) We now need an estimate on the number of sign changes of the solution u. To obtain it, we refer to the article of H. Matano [49] in which an estimate on the lap number of a solution of a parabolic problem is proved.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. We define its *lap number* l as the supremum of 0 and all $k \in \mathbb{N}$ with the property that there exist $k+1$ points $x_0 < \cdots < x_k$ such that

$$\forall 0 < i < k, \quad (g(x_{i+1}) - g(x_i))(g(x_i) - g(x_{i-1})) < 0.$$

We adapt the lemma of H. Matano [49] to get:

Lemma 1.7. *For any bounded solution $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ of (1.20):*

$$\partial_t V + \partial_u f(0,x)\partial_x V + F(\partial_x V, x) = \partial_x^2 V$$

with $V(0,\cdot) \in \mathcal{C}_0^\infty(\mathbb{R})$ having a finite lap number, the lap number of $V(t,\cdot)$ is uniformly bounded for all $t \geqslant 0$.

To do that, we just have to notice that $F(\partial_x V, x) = \tilde{F}(\partial_x V, x)\partial_x V$ with $\tilde{F}(\partial_x V, x)$. Since the number of sign changes of b is finite, $V(0,x)$ has a finite lap number. The lemma of H. Matano proves that $\forall t, \exists \xi_1^t, \ldots, \xi_m^t$ such that V is monotone on $]-\infty = \xi_0^t; \xi_1^t[, \ldots,]\xi_m^t; \xi_{m+1}^t = \infty[$. Therefore, the sign of u does not change on the same intervals. We deduce:

$$\begin{aligned}\|u(\cdot,t)\|_1 = \sum_{i=0}^m \left| \int_{\xi_i^t}^{\xi_{i+1}^t} u(x,t)\,\mathrm{d}x \right| &= \sum_{i=0}^m |V(\xi_{i+1}^t, t) - V(\xi_i^t, t)| \\ &\leqslant 2(m+1)\|V(t)\|_\infty \to 0.\end{aligned}$$

Therefore the theorem is proved under the assumptions:

$$\max\{\|w_{p^+}\|_\infty, \|w_{p^-}\|_\infty\} < r, \quad \|b\|_1 \sup_{[-r,r]} |\partial_u^2 F| \leqslant 1$$

with

$$p^+ = \min\{p : b \leqslant w_p\}, \quad p^- = \max\{p : b \geqslant w_p\}.$$

(4) Next, we show how to remove these assumptions. We define

$$A_p = \left\{ b \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} b = 0 \text{ and } \forall x, w_{-p}(x) \leqslant b(x) \leqslant w_p(x) \right\}.$$

We note $M_p = \max\{\|w_{-p}\|_\infty, \|w_p\|_\infty\}$. Hence, we have

$$\sup_{[-M_p,M_p]} |\partial_u^2 F| < \infty.$$

Let now $b \in A_p$ and $n = 2\|b\|_1 \sup_{[-M_p,M_p]} |\partial_v^2 F|$. Using $w_{-p} \leqslant 0 \leqslant w_p$, we have $b/n \in A_p$ and $\forall k \in \{1,\ldots,n\}$, $\frac{kb}{n} \in A_p$. The properties of the nonlinear semigroup show that A_p is stable under S^t, so we have for all t, $S^t(\frac{kb}{n}) \in A_p$.
By induction on k, we can prove the theorem for $\frac{kb}{n}$. Let P_k the property:

$$P_k : \lim_{t\to\infty} \left\| S^t\Big(\frac{kb}{n}\Big)\right\|_1 = 0$$

P_1: We have $b/n \in A_p$, $\|\frac{b}{n}\|_1 \sup_{[-M_p,M_p]} |\partial_u^2 F| = \frac{1}{2} < 1$. Using step (3), we deduce

$$\lim_{t\to\infty} \left\| S^t\Big(\frac{b}{n}\Big)\right\|_1 = 0.$$

P_k: Let assume that P_k with $k < n$ is true and let prove P_{k+1}. We have $S^t((k+1)\frac{b}{n}) \in A_p$. Moreover, the L^1-contraction property gives

$$\left\| S^t\left((k+1)\frac{b}{n}\right) - S^t\left(\frac{kb}{n}\right)\right\|_1 \leqslant \left\|\frac{b}{n}\right\|_1.$$

We deduce:

$$\left\| S^t\left((k+1)\frac{b}{n}\right)\right\|_1 \leqslant \left\| S^t\left(\frac{kb}{n}\right)\right\|_1 + \left\|\frac{b}{n}\right\|_1.$$

Since

$$\lim_{t\to\infty} \left\| S^t\left(\frac{kb}{n}\right)\right\|_1 = 0,$$

we have

$$\left\| S^t\left((k+1)\frac{b}{n}\right)\right\|_1 \sup_{[-M_p,M_p]} |\partial_u^2 F| < 1$$

for t large enough. Furthermore, $S^t((k+1)\frac{b}{n}) \in A_p$. Hence, we can use the conclusion of step (3) again to conclude the proof. □

1.5 Perspectives

In this paper we have proved the L^1-stability of the periodic stationary solutions of (1.1) in the one dimensional space case. The proof uses a dispersion inequality which is also verified in the multidimensional space case and the lemma of H. Matano (Lemma 1.7) about the number of sign changes of the solution of (1.1). But in the multidimensional space case, the lemma of H. Matano has no more sense. An idea to bypass it is to use Duhamel's formula, as done by D. Serre in [65]. In this purpose, we consider the linearized operator $L = \Delta - \operatorname{div}(\partial_u f(0,x)\cdot)$, and we write the equation in the form:

$$(\partial_t - L)u = -\operatorname{div}(F(u,x))$$

with $F(u,x) = f(u,x) - \partial_u f(0,x)u$. We note $\tilde{K}^t$ the kernel of the operator $\partial_t - L$ so that we obtain Duhamel's formula:

$$u(t) = \tilde{K}^t * b - \int_0^t \nabla_x \tilde{K}^{t-s} * F(u(s,\cdot),\cdot)\,\mathrm{d}s.$$

Taking L^1-norms:

$$u(t) \leqslant \|\tilde{K}^t * b\|_1 + \int_0^t \|\nabla_x \tilde{K}^{t-s}\|_1 \|F(u(s,\cdot),\cdot)\|_1\,\mathrm{d}s. \tag{1.23}$$

Moreover, we have $\partial_u F(0,\cdot) \equiv 0$, so we obtain $|F(u,\cdot)| \leqslant |u|^2$. Hence, dispersion inequality (1.11) gives

$$\|F(u(s,\cdot),\cdot)\|_1 \leqslant C_d^2 \frac{\|b\|_1^2}{s^{d/2}}.$$

To obtain an L^1-convergence theorem similar to Theorem 1.2, we can use estimates on the kernel $\tilde{K}^t$ and its derivative $\nabla_x \tilde{K}^t$. Some results on this kernel are given by M. Oh and K. Zumbrun in [57] and [58] when the space dimension is one. When the space dimension d is larger than 2, we can refer to [55] and [56] in which they obtain large-time estimates in L^q where $q \geqslant 2$, and when f is periodic in only one direction. But, until now, we have not large-time L^1-estimates for $d \geqslant 2$.

To conclude, we can see how estimates can give a theorem: if we obtain suitable estimates, we can bound all the term in (1.23) by $\|b\|_1^2$ as in [65] and conclude as D. Serre does by continuity of the limit: $l_0(b) = \lim_{t\to\infty} \|S^t b\|_1$.

Chapitre 2

Persistence of generalized roll-waves under viscous perturbation

ABSTRACT. The purpose of this article is to study the persistence of solution of a hyperbolic system under small viscous perturbation. Here, the solution of the hyperbolic system is supposed to be periodic: it is a periodic perturbation of a roll-wave. So, it has an infinity of shocks. The proof of the persistence is based on an expansion of the viscous solution and estimates on Green's functions.

2.1 Introduction

In this paper, we consider a one-dimensional system

$$u_t^\varepsilon + f(u^\varepsilon)_x = g(u^\varepsilon) + \varepsilon u_{xx}^\varepsilon \tag{2.1}$$

with a smooth flux $f : \mathbb{R}^n \to \mathbb{R}^n$ and a smooth function $g : \mathbb{R}^n \to \mathbb{R}^n$, which will be supposed linear in the main theorem. We assume that the corresponding system without viscosity

$$u_t + f(u)_x = g(u) \tag{2.2}$$

is strictly hyperbolic.

We consider a piecewise smooth function u which is a distributional solution of (2.2) on the domain $\mathbb{R} \times [0; T^*]$. We assume that u is periodic in the x variable, with a period L and that u has m noninteracting Lax shocks per period.

When g is a linear function, we show that u is a strong limit of solutions u^ε of (2.1) as $\varepsilon \to 0$. This work is of course motivated by the conjecture that the admissible solutions of (2.2) are strong limits of solutions of (2.1) with the same initial data.

In the case of scalar conservation laws, the proof of this conjecture uses the maximum principle [71], and in the case of special 2×2 systems, R. J. DiPerna proved it by a compensated compactness argument [23]. For the general case of shocks, there is a first paper of J. Goodman and Z. P. Xin which proves this conjecture for small amplitude Lax shocks [32] and an other one of A. Bressan and T. Yang which gives an estimate of the rate of convergence in the case of small total variation [12]. This conjecture is also proved for a single non-characteristic Lax shock or overcompressive shock by F. Rousset [63]. Here, we only consider Lax shocks but we have an infinity of shocks.

An other motivation of this work states in the study of roll-waves, in fluid mechanics or in general hyperbolic systems with source terms. Indeed, P. Noble proved the existence of roll-waves for this kind of system under assumptions on the source term [54]. Specifically, in the case of inviscid Saint Venant equations

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (g\cos\theta \dfrac{h^2}{2} + hu^2)_x = gh\sin\theta - c_f u^2, \end{cases}$$

one can prove that there exist roll-waves which are persistent under small perturbation [53]. So, there exist solutions of inviscid Saint Venant equations, near roll-waves. Here, the idea is to prove that there exists a family of solutions of the viscous Saint Venant system

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (g\cos\theta \dfrac{h^2}{2} + hu^2)_x = gh\sin\theta - c_f u^2 + \varepsilon(hu_x)_x, \end{cases} \tag{2.3}$$

which tends to a solution of inviscid system as ε goes to 0. We prove this result in the case of full viscosity, and with linear source term.

We can now give the full set of assumptions and formulate our main result. First, we suppose that

(H1) system (2.2) is strictly hyperbolic.

That means that there exist smooth matrices $P(u), D(u)$ such that

$$\mathrm{d}f(u) = P(u)D(u)P(u)^{-1}$$

where $D(u) = \mathrm{diag}(\lambda_1(u), \ldots, \lambda_n(u))$ is a diagonal matrix and $\lambda_i \neq \lambda_j$ for all $i \neq j$.

(H2) u is a distributional solution of (2.2) on $[0; T^*]$. Moreover, we suppose that u is piecewise smooth, periodic, and has m noninteracting and non-characteristic Lax shocks per period.

That means that u is smooth except at the points (x, t) of smooth curves $x = X_j(t) + iL, j = 1, \ldots, m, i \in \mathbb{Z}$ and that for all $j, k, t, |X_j(t) - X_k(t)| > 2r > 0$ (see Figure 2.1).

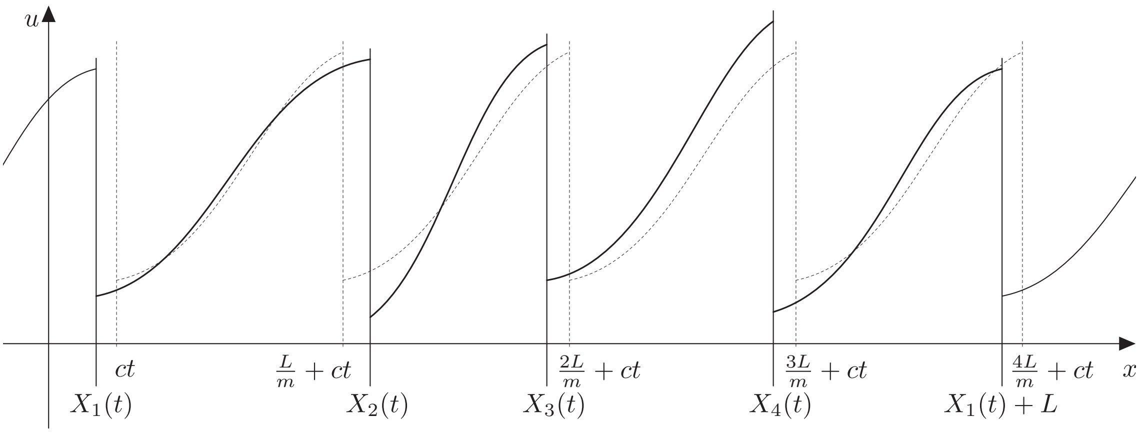


Figure 2.1: Allure of solution u over one period when u is scalar and $m = 4$. The periodic roll-wave is drawn in dotted line. The solution which checks our assumptions is represented by continuous line. One also placed the shocks for the two solutions.

Moreover, following limits are finite:

$$\partial_x^k u^{j\pm}(t) := \partial_x^k u(X_j(t) \pm 0, t) = \lim_{x \to X_j(t)^\pm} \partial_x^k u(x, t).$$

Since the shocks are non-characteristic k-Lax shocks, we have:

$$\lambda_1(u^{j-}) \leqslant \cdots \leqslant \lambda_{k-1}(u^{j-}) < X_j'(t) < \lambda_k(u^{j-}) \leqslant \cdots \leqslant \lambda_n(u^{j-}),$$

$$\lambda_1(u^{j+}) \leqslant \cdots \leqslant \lambda_k(u^{j+}) < X_j'(t) < \lambda_{k+1}(u^{j+}) \leqslant \cdots \leqslant \lambda_n(u^{j+}).$$

This assumption ensures the existence of at least one sonic point between two shocks.

We refer to [53] for the existence of such a solution in the case of Saint Venant equations. This result can be extended to general hyperbolic systems.

(H2') There exists a viscous profile for each shock for all $t \in [0;T^*]$.

More precisely, for all j, t, there exists $V^j(t)$ such that

$$V^j_{\xi\xi} - (f(V^j) - X'_j(t)V^j)_\xi = 0 \tag{2.4}$$

and

$$V^j(\pm\infty, t) = u^{j\pm}(t).$$

We will give more details on the properties of V^j in Section 2.2.2. Now, we only need to expose some assumption of linear stability. Consider for $\tau \leqslant T^*$, the operator

$$\mathcal{L}^j_\tau w = w_{zz} - (\mathrm{d}f(V^j(z,\tau)) - X'_j(\tau))w_z.$$

We assume that the viscous shock profiles are linearly stable. This assumption is equivalent to an Evans function criterion [74].

(H3) $\forall \tau \in [0;T^*], j = 1, \ldots, m, \mathcal{L}^j_\tau$ is such that $D^j_\tau(\lambda) \neq 0 \quad \forall \lambda, \Re\lambda \geqslant 0, \lambda \neq 0$, and $D^{j\prime}_\tau(0) \neq 0$, where D^j_τ is the Evans function of $\mathcal{L}^j_\tau$.

We can now state our main theorem:

Theorem 2.1. *Under assumptions* **(H1)–(H2)–(H2')–(H3)** *and if g is linear, $g(u) = \kappa u$, for all $\varepsilon > 0$, there exists a solution u^ε of* (2.1) *on* $[0;T^*]$ *such that*

$$\|u^\varepsilon(t=0) - u(t=0)\|_{L^1(0;L)} \to 0 \qquad \text{as } \varepsilon \to 0, \tag{2.5}$$

and we have the convergences

$$\|u^\varepsilon - u\|_{L^\infty([0;T^*],L^1(0;L))} \to 0, \qquad \text{as } \varepsilon \to 0.$$

And for any $\eta \in (0,1)$,

$$\sup_{0 \leqslant t \leqslant T^*, |x - X_j(t)| \geqslant \varepsilon^\eta} |u^\varepsilon(x,t) - u(x,t)| \to 0, \qquad \text{as } \varepsilon \to 0.$$

The proof of this theorem is done in three steps: construction of an approximate solution (which gives us an expansion of u^ε in ε), estimates on the semigroup generated by linearized operator around this approximate solution and a Banach fixed point argument to deal with the full problem.

The paper is organized as follows. In Section 2.2, we build an approximate solution u^ε_{app} of the full problem (2.1) close to u, solution of (2.2) up to order 2 with respect to ε. This is done separating slow parts where u^ε_{app} is close to u and shock parts where $u^\varepsilon_{app}|_{X_j\pm\varepsilon^\gamma}$ is close to V_j. More precisely, one expands u^ε in the slow part as

$$u^\varepsilon(x,t) = u(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + o(\varepsilon^2)$$

where u is the solution of (2.2) and u_i are solutions of the linearized equation of (2.2) around u, which is well-posed thanks to assumption **(H3)**. In shock parts, the expansion at shock j is

$$u^\varepsilon(x,t) = V^j(\xi^j(x,t,\varepsilon),t) + \varepsilon V_1^j(\xi^j(x,t,\varepsilon),t) + \varepsilon^2 V_2^j(\xi^j(x,t,\varepsilon),t) + o(\varepsilon^2)$$

where the stretched variable is $\xi^j(x,t,\varepsilon) = \frac{x-X_j(t)}{\varepsilon} + \delta^j(t)$, V^j is solution of viscous equation (2.4) and $V_i^j, i = 1,2$, are solutions of the linearized equation of (2.4) around V^j. Moreover, the functions u_i, V_i^j are related by matching conditions which ensure regularity on the approximate solution u^ε_{app}, built by convex combination of the expansions:

$$u^\varepsilon_{app} = \sum_{j=1}^m \mu\left(\frac{x-X_j(t)}{\varepsilon^\gamma}\right) I^{j\varepsilon}(x,t) + \left(1 - \sum_{j=1}^m \mu\left(\frac{x-X_j(t)}{\varepsilon^\gamma}\right)\right) O^\varepsilon(x,t) + d^\varepsilon(x,t).$$

where

$$\mu(x) = \begin{cases} 0 \text{ if } |x| > 2, \\ 1 \text{ if } |x| < 1, \end{cases}$$

$$\begin{aligned} I^{j\varepsilon}(x,t) &= V^j(\xi^j(x,t,\varepsilon),t) + \varepsilon V_1^j(\xi^j(x,t,\varepsilon),t) + \varepsilon^2 V_2^j(\xi^j(x,t,\varepsilon),t), \\ O^\varepsilon(x,t) &= u(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t), \end{aligned}$$

and d^ε verifies a heat equation with source term.

With this construction, we prove the theorem:

Theorem 2.2. *There exists an approximate solution u^ε_{app} of (2.1) defined on $[0;T^*]$. If φ is a smooth change of variable which fixes the shocks ($\forall t,i,j, \varphi((j-1)\frac{L}{m}+iL,t) = X_j(t)+iL$), and $\tilde u^\varepsilon_{app}(z,t) = u^\varepsilon_{app}(\varphi(z,t),t)$, then $\tilde u^\varepsilon_{app}$ verifies the equation*

$$(u^\varepsilon_{app})_t + f(u^\varepsilon_{app})_x - \varepsilon(u^\varepsilon_{app})_{xx} - \kappa u^\varepsilon_{app} = -R^\varepsilon_x(x,t)$$

with the following estimates on $\tilde R^\varepsilon$, where $\tilde R^\varepsilon(z,t) = R^\varepsilon(\varphi(z,t),t)$,

$$\|\tilde R^\varepsilon\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma}, \|\tilde R^\varepsilon_t\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma-1/2}, \quad \|\tilde R^\varepsilon_{tt}\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma-1}, \tag{2.6}$$

$$\|\tilde R^\varepsilon_z\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma-1}, \quad \|\tilde R^\varepsilon_{tz}\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma-3/2}, \tag{2.7}$$

$$\|\tilde R^\varepsilon_{zz}\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma-2}. \tag{2.8}$$

Here, u^ε_{app} is constructed as a perturbation of u going to order 2, which allows us to have estimates on $\tilde{R}^\varepsilon_{zz}$ in L^1 in $\varepsilon^{3\gamma-2}$, $3\gamma-2>0$. This property will be useful to prove the convergence of $u^\varepsilon_{app}-u^\varepsilon$ to 0.

In Section 2.3, we linearize (2.1) in the neighbourhood of the approximate solution u^ε_{app} and we compute estimates on the Green's function. To do so, we use the method of iterative construction of the Green's function, which was first used by E. Grenier and F. Rousset in [34]. So, we consider approximations of the Green's functions in neighbourhood of the shocks (given by K. Zumbrun and P. Howard in [74]) and we build our own approximation far away from the shock, using the characteristic curves.

The last section is dedicated to the proof of theorem

Theorem 2.3. *Under assumptions* **(H1)–(H2)–(H2')–(H3)** *and for linear g, for all ε, there exists u^ε solution of* (2.1) *on* $(0;T^*)$ *such that*

$$u^\varepsilon(t=0)\equiv u^\varepsilon_{app}(t=0).$$

And this u^ε verifies the convergences:

$$\|u^\varepsilon-u^\varepsilon_{app}\|_{L^\infty((0;T^*)\times\mathbb{R})}\to 0,$$

$$\|u^\varepsilon-u^\varepsilon_{app}\|_{L^\infty((0;T^*),L^1(\mathbb{R}))}\to 0$$

when ε goes to zero.

This is done using standard arguments for parabolic problems. Indeed, we combine estimates on $\tilde{q}^\varepsilon$, and estimate on the Green's function to obtain estimates on $u^\varepsilon-u^\varepsilon_{app}$, and its derivatives, depending on ε and uniform in time for ε small enough. Then, using the convergence of u^ε_{app} to u, we immediately deduce Theorem 2.1.

2.2 Construction of the approximate solution

The purpose of this section is to prove Theorem 2.2 on the existence of the approximate solution u^ε_{app} of (2.1). In a first step, we compute formally this approximate solution using outer and inner expansions of order 2. Indeed, in slow part, where ∇u is bounded, the solution u^ε of (2.1) may be approximated by truncation of the formal series

$$u^\varepsilon(x,t)\sim O^\varepsilon(x,t)=u(x,t)+\varepsilon u_1(x,t)+\varepsilon^2 u_2(x,t)$$

where u is the solution of (2.2) we want to approach. Similarly, near the shocks j, we search for u^{ε}_{app} with the inner expansion

$$I^{j\varepsilon}(x,t) = V^j(\xi^j(x,t,\varepsilon),t) + \varepsilon V^j_1(\xi^j(x,t,\varepsilon),t) + \varepsilon^2 V^j_2(\xi^j(x,t,\varepsilon),t)$$

where $\xi^j(x,t,\varepsilon) = \frac{x-X_j(t)}{\varepsilon} + \delta^j_0(t) + \varepsilon\delta^j_1(t)$ is the stretched variable and V^j is the viscous shock profile, solution of (2.4). We match this expansion by continuity of u^{ε}_{app} and its spatial derivatives.

In this section, we formally substitute these expansions in (2.1) to find equations satisfied by u_i and V^j_i, $i = 1, 2, j = 1, \ldots, m$, and matching conditions. Then, we prove the existence of the u_i and V^j_i on $(0;T^*)$. Furthermore, we give rigorous estimates on the error terms. We can remark here that we search for an approximation of order 2, this will be useful to obtain good estimates on the second derivatives of the error term.

2.2.1 Formal calculation and derivation of the equations

Substituting O^{ε} into (2.1) and identifying the power of ε in the expressions, we get for $x \neq X_j(t)$:

$$\begin{aligned}
&\mathcal{O}(\varepsilon^0) : u_t + (f(u))_x - g(u) = 0,\\
&\mathcal{O}(\varepsilon^1) : u_{1,t} + (\mathrm{d}f(u)\cdot u_1)_x - \mathrm{d}g(u)\cdot u_1 = u_{xx},\\
&\mathcal{O}(\varepsilon^2) : u_{2,t} + (\mathrm{d}f(u)\cdot u_2)_x - \mathrm{d}g(u)\cdot u_2 = u_{1xx} - \frac{1}{2}(\mathrm{d}^2 f(u)\cdot(u_1,u_1))_x\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}\,\mathrm{d}^2 g(u)\cdot(u_1,u_1).
\end{aligned}$$

A similar calculation for $I^{j\varepsilon}$ yields the set of equations:

$$\begin{aligned}
\mathcal{O}(\varepsilon^{-1}) &: V^j_{\xi\xi} - (f(V^j) - X'_j V^j)_\xi = 0,\\
\mathcal{O}(\varepsilon^{0}) &: V^j_{1\xi\xi} - ((\mathrm{d}f(V^j) - X'_j)\cdot V^j_1)_\xi = V^j_t + V^j_\xi \delta^j_{0t} - g(V^j),\\
\mathcal{O}(\varepsilon^{1}) &: V^j_{2\xi\xi} - ((\mathrm{d}f(V^j) - X'_j)\cdot V^j_2)_\xi = V^j_{1t} + V^j_{1\xi}\delta^j_{0t} + V^j_\xi\delta^j_{1t} + \frac{1}{2}(\mathrm{d}^2 f(V^j)\cdot(V^j_1,V^j_1))_\xi\\
&\qquad - \mathrm{d}g(V^j)\cdot V^j_1.
\end{aligned}$$

We remark that the equations for u_i are hyperbolic equations: the first one is (2.2), so nonlinear, and the others are the linearization of (2.2) around u. Similarly, the equations for the shock profiles are ordinary equations: nonlinear for V^j, we recognize (2.4), and its linearization around V^j for V^j_1 and V^j_2. To maximize the order of the approximation, we couple these equations with boundary conditions, connecting u_i and V_i^j. First, we note

$$\partial_x^k u_i^{j\pm}(t) := \partial_x^k u_i(X_j(t) \pm 0, t) = \lim_{x \to X_j(t)^\pm} \partial_x^k u_i(x,t), \quad i = 1, 2.$$

Then, we rewrite O^ε and I^ε with the variable ξ, in a vicinity of shock j, and we ask the two functions to coincide as ε goes to 0. Therefore, we make Taylor expansion of order 2 with respect to ε. For example, for $\xi > 0$, large enough,

$$\begin{aligned} O^\varepsilon(X_j(t) + \varepsilon(\xi - \delta_0^j(t) - \varepsilon\delta_1^j(t)), t) &= u^{j+}(t) + \varepsilon\big(u_1^{j+}(t) + u_x^{j+}(\xi - \delta_0^j)\big) \\ &+ \frac{\varepsilon^2}{2}\big(u_2^{j+}(t) + 2u_{1x}^{j+}(t)(\xi - \delta_0^j) - 2u_x^{j+}\delta_1^j + u_{xx}^{j+}(t)(\xi - \delta_0^j)^2\big) + o(\varepsilon^2) \end{aligned}$$

and

$$I^\varepsilon(X_j(t) + \varepsilon(\xi - \delta_0^j(t) - \varepsilon\delta_1^j(t)), t) = V^j(\xi, t) + \varepsilon V_1^j(\xi, t) + \varepsilon^2 V_2^j(\xi, t) + o(\varepsilon^2)$$

Identifying the terms of same order on ε, we get as $\xi \to \pm\infty$:

$$V^j(\pm\infty, t) = u^{j\pm}(t), \tag{2.9}$$

$$V_1^j(\xi, t) = u_1^{j\pm}(t) + u_x^{j\pm}(t)(\xi - \delta_0^j(t)) + o(1), \tag{2.10}$$

$$V_2^j(\xi, t) = u_2^{j\pm}(t) + u_{1x}^{j\pm}(t)(\xi - \delta_0^j(t)) + \frac{1}{2}u_{xx}^{j\pm}(t)(\xi - \delta_0^j(t))^2 - u_x^{j\pm}(t)\delta_1^j(t) + o(1). \tag{2.11}$$

For more details on the computation of these conditions, we refer to [26].

2.2.2 Existence of solutions of the outer and inner problems

In this section, we show that the solutions u_i and V_i^j of the previous equations exist under assumption **(H3)** on the spectral stability of the viscous shock profile. We first remark that the leading-order outer function u is exactly the solution of (2.2) which we want to approximate. Therefore, we first prove the existence of the V^j, and then we prove the existence of u_1 and all the V_1^j. Simultaneously, we prove the existence of the δ_0^j. Similarly, we prove the existence of u_2, V_2^j and δ_1^j.

Construction at order 0

In this section, we deal with the existence of u, V^j which satisfy equations (2.2)-(2.4) and matching condition (2.9). The existence of u is exactly assumption **(H2)**.

Since u is a distributional solution, u verifies Rankine-Hugoniot conditions at each shock j:

$$f(u^{j+}) - f(u^{j-}) = X_j'(t)(u^{j+} - u^{j-}).$$

As said in assumption **(H2')**, we also assume the existence of the viscous shock profile V^j which verifies

$$V^j_{\xi\xi} - (f(V^j) - X_j'(t)V^j)_\xi = 0 \tag{2.12}$$

and the asymptotic conditions:

$$V^j(\pm\infty, t) = u^{j\pm}(t).$$

We refer to N. Kopell and L. N. Howard [44] for the existence of such a profile under smallness assumption on the amplitude of the shock. In our case, we assume this existence, and the structure of the shock gives the convergence rate of the profile and its derivatives as $\xi \to \pm\infty$. Indeed, since shocks are Lax-shocks, u^{j+} and u^{j-} are hyperbolic rest points for the ordinary differential equation (2.12), and for some $\omega > 0$, for any $\alpha \in \mathbb{N}$, we have

$$\begin{aligned} |\partial_t^\alpha V^j(\xi,t) - \partial_t^\alpha u^{j\pm}(t)| &\leqslant e^{-\omega|\xi|}, \quad \forall \xi \in \mathbb{R}, &(2.13)\\ |\partial_\xi^\alpha V^j(\xi,t)| &\leqslant e^{-\omega|\xi|}, \quad \forall \xi \in \mathbb{R}. &(2.14)\end{aligned}$$

Construction at order 1: existence of V_1^j and u_1

In this section, we prove the existence of u_1, V_1^j, δ_0^j on $(0;T^*)$ such that

$$\begin{cases} u_{1,t} + (\mathrm{d}f(u)\cdot u_1)_x - \mathrm{d}g(u)\cdot u_1 = u_{xx}, & (2.15)\\ V^j_{1\xi\xi} - ((\mathrm{d}f(V^j) - X_j')\cdot V_1^j)_\xi = V_t^j + V_\xi^j \delta_{0t}^j - g(V^j), & (2.16)\\ V_1^j(\xi,t) = u_1^{j\pm}(t) + u_x^{j\pm}(t)(\xi - \delta_0^j(t)) + o(1), \quad \xi \to \pm\infty. & (2.17)\end{cases}$$

We first remark that these equations are linear. As in [63], it is convenient to deal with bounded solutions. Therefore, we write

$$U_1^j = V_1^j - D_1^j$$

where D_1^j is a smooth function such that:

$$D_1^j = \begin{cases} \xi u_x^{j-}(t) & \text{if } \xi < -1,\\ \xi u_x^{j+}(t) & \text{if } \xi > 1.\end{cases}$$

Consequently, U_1^j solves:

$$\begin{cases} U_{1\xi\xi}^j - ((\mathrm{d}f(V^j) - X_j') \cdot U_1^j)_\xi = \delta_{0t}^j V_\xi^j + h^j(\xi, t), & (2.18) \\ U_1^j(\pm\infty, t) = u_1^{j\pm}(t) - \delta_0^j(t) u_x^{j\pm}(t), & (2.19) \end{cases}$$

with

$$h^j(\xi, t) = -D_{1\xi\xi}^j + V_t^j + ((\mathrm{d}f(V^j) - X_j')D_1^j)_\xi - g(V^j).$$

From estimates (2.13), (2.14), we deduce that h satisfies:

$$h^j(\xi, t) = \frac{\mathrm{d}}{\mathrm{d}t} u^{j\pm}(t) + (\mathrm{d}f(u^{j\pm}) - X_j') u_x^{j\pm}(t) - g(u^{j\pm}) + O(e^{-\alpha|\xi|}), \alpha > 0.$$

And, since u is a smooth solution of (2.2), we have $h \in L^1(\mathbb{R})$ and: $h^j(\xi, t) = O(e^{-\alpha|\xi|})$. Integrating (2.18) with respect to ξ yields

$$U_{1\xi}^j - (\mathrm{d}f(V^j) - X_j')U_1^j = \delta_{0t}^j V^j + \int_0^\xi h^j(\eta, t)\,\mathrm{d}\eta + C^j(t) \tag{2.20}$$

where $C^j(t)$ is a constant, only depending on t.

Let us solve the problem (2.15)-(2.20) with matching condition (2.19). Following [63], we construct the solution of this system in two steps. First, for all j, we fix t and δ_0^j, we find U_1^j solution of (2.20) with finite limits at $\pm\infty$. Since these limits are explicit and only depends on t, δ_0^j, and C^j, we use the matching condition (2.19) to rewrite (2.15) as a hyperbolic boundary value problem where u_1 and δ_0^j are the only unknowns. After solving this system, we use the previous construction to obtain U_1^j solution of (2.20) with matching conditions (2.19).

So, we fix t and δ_0^j for all j. With exactly the same arguments as in [63], we prove the existence of U_1^j for all j. Hence, using assumption **(H3)** on the viscous shock profile and theory of Fredholm operators, we show that U_1^j exists and the limits satisfy:

$$\lim_{\xi \to \pm\infty} U_1^j(\xi, t) = -(\mathrm{d}f(u^{j\pm}) - X_j')^{-1}(\delta_{0t}^j u^{j\pm} + H^{j\pm} + C^j)$$

where $H^{j\pm} = \int_0^{\pm\infty} h(\eta, t)\,\mathrm{d}\eta$.

We now use matching conditions (2.19) to eliminate C^j in these relations. Indeed, we have

$$(\mathrm{d}f(u^{j+}) - X_j')(u_1^{j+} - \delta_0^j u_x^{j+}) = -(\delta_{0t}^j u^{j+} + H^{j+} + C^j),$$

$$(\mathrm{d}f(u^{j-}) - X_j')(u_1^{j-} - \delta_0^j u_x^{j-}) = -(\delta_{0t}^j u^{j-} + H^{j-} + C^j),$$

and their difference is

$$A^{j+}u_1^{j+} - A^{j-}u_1^{j-} + \delta_{0t}^j(u^{j+} - u^{j-}) = \delta_0^j(A^{j+}u_x^{j+} - A^{j-}u_x^{j-}) - (H^{j+} - H^{j-}) \quad (2.21)$$

where $A^{j\pm} = \mathrm{d}f(u^{j\pm}) - X_j'(t)$.

Now, we have to solve (2.15), (2.21). In order to find a solution of this system, we rewrite it by fixing the shocks. Since the shocks do not interact, we can define a change of variable Z (see Figure 2.2) which is bijective, continuous in (x,t), piecewise linear in x and piecewise smooth:

$$Z(t,x) = \frac{x - X_j(t)}{X_{j+1}(t) - X_j(t)}\frac{L}{m} + (j-1)\frac{L}{m} \qquad \text{if } x \in [X_j(t); X_{j+1}(t)], i = 1, \ldots, m.$$

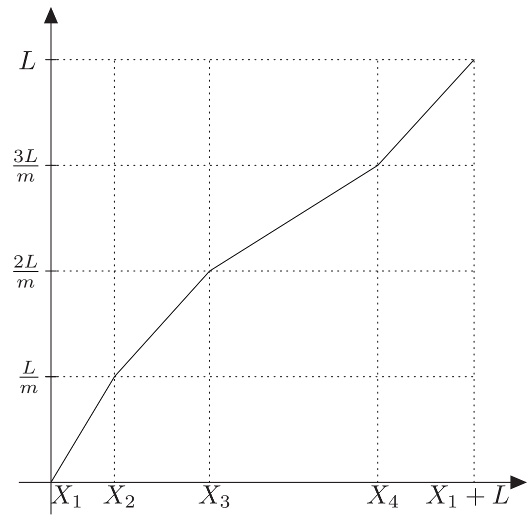


Figure 2.2: Example of the change of variable Z in the case $m = 4$, for some t.

We also define v_1 by

$$u_1(x,t) = v_1(Z(t,x),t).$$

It follows from these definitions that v_1 solves

$$v_{1t} + (Z_x\,\mathrm{d}f(u) + Z_t)(Z^{-1}(t,z),t)v_{1z} + \bar{h}(z,t)v_1 - \tilde{h}(z,t) = 0 \quad (2.22)$$

where $z \mapsto x = Z^{-1}(t,z)$ is the inverse of $x \mapsto z = Z(t,x)$, and, $\bar{h}$ and $\tilde{h}$ only depend on Z, u, u_x, and u_{xx}.

We now use the fact that $\mathrm{d}f(u)$ is diagonalizable, $\mathrm{d}f(u) = P(u)^{-1}D(u)P(u)$ so

$$(Z_x\,\mathrm{d}f(u) + Z_t)(Z^{-1}(t,z),t) = \tilde{P}(z,t)^{-1}\tilde{D}(z,t)\tilde{P}(z,t).$$

Since zeroth order terms do not play any role in the wellposedness issue, we consider the simplified system

$$\begin{cases} w_{1t} + \tilde{D}(z,t)w_{1z} = \tilde{k}(z,t), & (2.23)\\ A^{j+}(P^{j+})^{-1}w_1^{j+} - A^{j-}(P^{j-})^{-1}w_1^{j-} + \delta_{0t}^j(u^{j+} - u^{j-}) = l^j(t), & (2.24)\end{cases}$$

with $\tilde{k}$ and l^j known functions. Therefore, we have to solve this system on $[0;L]$ under periodic boundary conditions. Since 0 and L correspond to the same shock, the periodic boundary conditions are in fact the shock conditions (2.24) for $j = 1$.

Equation (2.23) is a linear transport equation on $w_{1i}, i = 1, \ldots, n$. Since, generically, the existence of u smooth on $[0;T^*]$ ensures that the characteristics do not intersect on $[0;T^*]$, they can be used to build $w_{1,i}$ smooth between shocks, using the initial condition. So, it suffices to verify that the conditions (2.24) at the shocks are well-posed. We must therefore count the incoming and outgoing information at the shock. As we can see in Figure 2.3, for $i < k$, the incoming characteristics come from the right, so we obtain the value of w_{1i}^{j+}. For $i > k$, the incoming characteristics come from the left, so we get w_{1i}^{j-}. And for $i = k$, the sign of the eigenvalue change between two shocks: negative on the right of a shock and positive on the left, so, using again characteristic construction, w_{1k} is defined on the whole interval delimited by the shocks: we obtain w_{1k}^{j+} and w_{1k}^{j-}.By this method we have built w_{1i} on the right or left side of each shock. We now use the boundary conditions (2.24) to obtain all the components of w_1^j, and δ_{0t}^j.

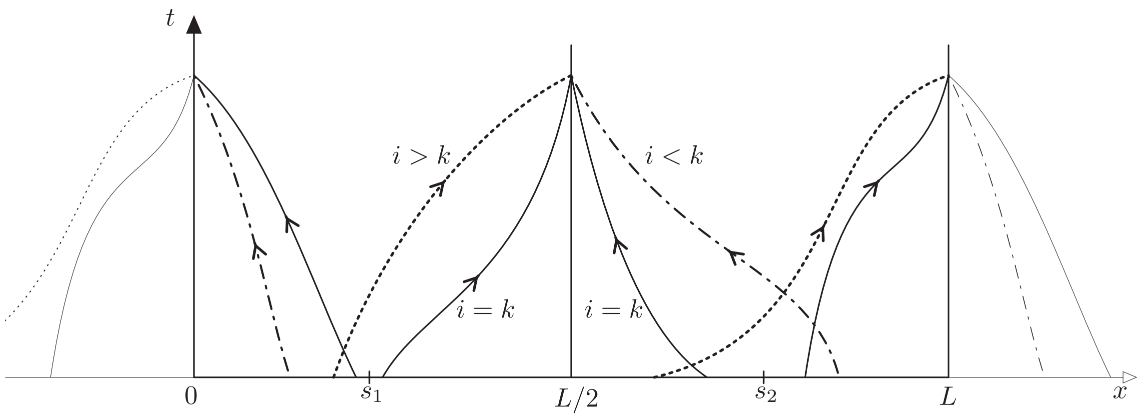


Figure 2.3: Characteristic curves between two shocks, example with $m = 2$. We also plot s_1 and s_2 which are sonic points for k-th eigenvalue, that means $\lambda_k(s_j) = 0$.

Indeed, if we note by r_i the i-th eigenvector of $\mathrm{d}f(u)$, and $w_1 = \tilde{P}\sum_i a_i r_i$, then a_i^{j+} is known for $i \leqslant k$ and a_i^{j-} for $i \geqslant k$ by our construction and we rewrite (2.24) as

$$\sum_{i=1}^{n} A^{j+} a_i^{j+} r_i^{j+} - \sum_{i=1}^{n} A^{j-} a_i^{j-} r_i^{j-} + \delta_{0t}^j (u^{j+} - u^{j-}) = l^j(t)$$

or equivalently

$$\sum_{i=1}^{n} (\lambda_i^{j+} - X_j') a_i^{j+} r_i^{j+} - \sum_{i=1}^{n} (\lambda_i^{j-} - X_j') a_i^{j-} r_i^{j-} + \delta_{0t}^j (u^{j+} - u^{j-}) = l^j(t).$$

This yields the linear system on the unknowns a_i^{j+} for $i > k$, a_i^{j-} for $i < k$, and δ_{0t}^j

$$\begin{aligned}\sum_{i>k} (\lambda_i^{j+} - X_j') a_i^{j+} r_i^{j+} - \sum_{i<k} (\lambda_i^{j-} - X_j') a_i^{j-} r_i^{j-} + \delta_{0t}^j (u^{j+} - u^{j-}) \\ = l^j(t) - \sum_{i\leqslant k} (\lambda_i^{j+} - X_j') a_i^{j+} r_i^{j+} + \sum_{i\geqslant k} (\lambda_i^{j-} - X_j') a_i^{j-} r_i^{j-}.\end{aligned}$$

These equations have unique solutions if and only if the system obtained is invertible for all t, that is the Majda-Liu condition:

$$\forall j = 1, \ldots, m, \quad \det(r_1^{j-}, \ldots, r_{k-1}^{j-}, u^{j+} - u^{j-}, r_{k+1}^{j+}, \ldots, r_n^{j+}) \neq 0.$$

Using [75], our assumption **(H3)** implies Majda-Liu condition. To finish the construction of the approximate solution, we use again the characteristics. By this way, we have built a solution on the whole space $\mathbb{R}$. To ensure the regularity of the solution far away from the shocks, we only need suitable compatibility conditions on the initial data.

Finally, we have proved the existence of u_1 and δ_{0t}^j for all j and for $0 \leqslant t \leqslant T^*$. The previous construction give us V_1^j for all j. We can then apply the same method to obtain the existence of V_2^j, δ_{1t}^j and u_2, since the linear system has the same terms of maximal order.

Remarks 2.1.

- Since the construction of viscous shock profile only depends on the shock, we can use the previous construction even if u is not periodic. However, we will see in the following that the periodicity of u allows us first to obtain bounds on u_1 and secondly to build the Green's function in Section 2.3.
- In this section, we only assume that g is a smooth function. We will see in the following that we need to integrate the equation, so we will assume that g is linear.

  we also have estimates on the error term for nonlinear g.

2.2.3 Construction of the approximate solution

We complete the construction of an approximate solution of equation (2.1). First, we define a smooth function μ such that:

$$\mu(x) = \begin{cases} 0 \text{ if } |x| > 2, \\ 1 \text{ if } |x| < 1. \end{cases}$$

Then, the approximate solution u^ε_{app} is defined as

$$u^\varepsilon_{app} = \sum_{j=1}^{m} \mu\left(\frac{x - X_j(t)}{\varepsilon^\gamma}\right) I^{j\varepsilon}(x,t) + \left(1 - \sum_{j=1}^{m} \mu\left(\frac{x - X_j(t)}{\varepsilon^\gamma}\right)\right) O^\varepsilon(x,t) + d^\varepsilon(x,t)$$

where d^ε verifies the parabolic equation

$$d^\varepsilon_t - \varepsilon d^\varepsilon_{xx} - \kappa d^\varepsilon = -q^\varepsilon \tag{2.25}$$

with initial data

$$d^\varepsilon(x,0) = 0,$$

and $q^\varepsilon(x,t) = \sum_{i=1}^{3} q^\varepsilon_i(x,t)$ is an error term given by

$$\begin{aligned}
q^\varepsilon_1(x,t) =&(1-\mu^j)\big[\big(f(O^\varepsilon) - f(u) - \varepsilon\, \mathrm{d}f(u)\cdot u_1 - \varepsilon^2\, \mathrm{d}f(u)\cdot u_2 - \frac{\varepsilon^2}{2}\, \mathrm{d}^2 f(u)\cdot(u_1,u_1)\big)_x \\
&- \kappa\big(O^\varepsilon - u - \varepsilon u_1 - \varepsilon^2 u_2\big) \\
&- \varepsilon^3 u_{2xx}\big],\\
q^\varepsilon_2(x,t) =&\mu^j[(f(I^{j\varepsilon}) - f(V^j) - \varepsilon\, \mathrm{d}f(V^j)\cdot V^j_1 - \varepsilon^2\, \mathrm{d}f(V^j)\cdot V^j_2 - \frac{\varepsilon^2}{2}\, \mathrm{d}^2 f(V^j)\cdot(V^j_1, V^j_1))_x \\
&- \kappa(I^{j\varepsilon} - V^j - \varepsilon V^j_1) \\
&+ \varepsilon^2(\delta_{1t} V^j_{1\xi} + V^j_{2t} + \delta' V^j_{2\xi})],\\
q^\varepsilon_3(x,t) =&\mu^j_t(I^{j\varepsilon} - O^\varepsilon) - \varepsilon\mu^j_{xx}(I^{j\varepsilon} - O^\varepsilon) - 2\varepsilon\mu^j_x(I^{j\varepsilon} - O^\varepsilon)_x + \mu^j_x(f(I^{j\varepsilon}) - f(O^\varepsilon)) \\
&+ f(\mu^j I^{j\varepsilon} + (1-\mu^j)O^\varepsilon)_x - (\mu^j f(I^{j\varepsilon}) + (1-\mu^j) f(O^\varepsilon))_x,
\end{aligned}$$

and $\mu^j = \mu\left(\frac{x - X_j(t)}{\varepsilon^\gamma}\right)$.

Then, $u^\varepsilon_{app} - d^\varepsilon$ verifies

$$(u^\varepsilon_{app} - d^\varepsilon)_t + f(u^\varepsilon_{app} - d^\varepsilon)_x - \varepsilon(u^\varepsilon_{app} - d^\varepsilon)_{xx} - \kappa(u^\varepsilon_{app} - d^\varepsilon) = q^\varepsilon$$

and u^ε_{app} verifies

$$(u^\varepsilon_{app})_t + f(u^\varepsilon_{app})_x - \varepsilon(u^\varepsilon_{app})_{xx} - \kappa u^\varepsilon_{app} = -R^\varepsilon_x,$$

where $R^\varepsilon = f(u^\varepsilon_{app}) - f(u^\varepsilon_{app} - d^\varepsilon)$.

We now want to prove that u^ε_{app} is a good approximation of u^ε, that means $u^\varepsilon - u^\varepsilon_{app} \to 0$ when $\varepsilon \to 0$. Therefore, we define $w^\varepsilon_x = u^\varepsilon - u^\varepsilon_{app}$ where w solves:

$$w_t + \mathrm{d}f(u^\varepsilon_{app}) \cdot w_x - \varepsilon w_{xx} - \kappa w = -R^\varepsilon - Q_1(u^\varepsilon_{app}, w_x) \tag{2.26}$$

with

$$Q_1(u^\varepsilon_{app}, w_x) = f(w_x + u^\varepsilon_{app}) - f(u^\varepsilon_{app}) - \mathrm{d}f(u^\varepsilon_{app}) \cdot w$$

which is at least quadratic term in w_x.

Since $(u^\varepsilon_{app})_x$ is unbounded as ε goes to zero, we avoid the singularity by integrating the equation on w_x.

2.2.4 Estimates on the error term

To end with the proof of Theorem 2.2, it remains to compute the estimates on the error term R^ε, which only depends on d and u^ε_{app}. So, the first step is to compute estimates on q^ε. As in [32], we can estimate the support of functions q^ε_i:

$$\begin{aligned}
\operatorname{supp}(q^\varepsilon_1) &\subset \{(x,t) : |x - X_j(t)| \geqslant \varepsilon^\gamma\},\\
\operatorname{supp}(q^\varepsilon_2) &\subset \{(x,t) : |x - X_j(t)| \leqslant 2\varepsilon^\gamma\},\\
\operatorname{supp}(q^\varepsilon_3) &\subset \{(x,t) : \varepsilon^\gamma \leqslant |x - X_j(t)| \leqslant 2\varepsilon^\gamma\}.
\end{aligned}$$

To obtain estimates on q^ε_i and their derivatives, we first need to fix the shocks by a smooth change of variable. This manipulation cannot be avoided for the estimates on $q^\varepsilon_{it}, q^\varepsilon_{itt}$. So, we define

$$\varphi(z,t) = z + \sum_{j=1}^{m} \alpha_j(z,t) \left(X_j(t) - (j-1)\frac{L}{m} - \varepsilon\delta^j(t) \right) \tag{2.27}$$

where $\alpha_j(\cdot, t)$ are smooth functions, such that $\sum_j \alpha_j \equiv 1$, φ is increasing, $\varphi_z > 0$ and $\alpha_j(\cdot,t) \equiv 1$ on a neighbourhood $[(j-1)\frac{L}{m} - r; (j-1)\frac{L}{m} + r]$ of $(j-1)\frac{L}{m}$. We recall that in assumption **(H2)** we have supposed that $|X_{j+1} - X_j| > 2r$, which ensures the existence of such a φ.

With the notations

$$\tilde{w}(z,t) = w(\varphi(z,t),t), \tilde{u}^\varepsilon_{app}(z,t) = u^\varepsilon_{app}(\varphi(z,t),t),$$

$$\tilde{q}^\varepsilon(z,t) = q^\varepsilon(\varphi(z,t),t), \tilde{R}^\varepsilon(z,t) = R^\varepsilon(\varphi(z,t),t),$$

equation (2.26) becomes:

$$\tilde{w}_t + \frac{1}{\varphi_z}\left(\mathrm{d}f(\tilde{u}^\varepsilon_{app}) - \varphi_t + \varepsilon\frac{\varphi_{zz}}{\varphi_z^2}\right)\cdot\tilde{w}_z - \varepsilon\frac{1}{\varphi_z^2}\tilde{w}_{zz} - \kappa\tilde{w} = -\tilde{R}^\varepsilon - Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z). \tag{2.28}$$

Similarly, $\tilde{d}^\varepsilon(z,t) = d^\varepsilon(\varphi(z,t),t)$ verifies the equation

$$\tilde{d}^\varepsilon_t + \left(\varepsilon\frac{\varphi_{zz}}{\varphi_z^3} - \frac{\varphi_t}{\varphi_z}\right)\tilde{d}^\varepsilon_z - \varepsilon\frac{1}{\varphi_z^2}\tilde{d}^\varepsilon_{zz} - \kappa\tilde{d}^\varepsilon = -\tilde{q}^\varepsilon \tag{2.29}$$

and the following lemma gives estimates on $\tilde{q}^\varepsilon$:

Lemma 2.1. *There exists a positive constant C independent of ε such that*

$$\|\tilde{q}^\varepsilon\|_{L^\infty}, \|\tilde{q}^\varepsilon_t\|_{L^\infty}, \|\tilde{q}^\varepsilon_{tt}\|_{L^\infty} \leqslant C\varepsilon^{2\gamma}, \tag{2.30}$$

$$\|\tilde{q}^\varepsilon\|_{L^1(0;L)}, \|\tilde{q}^\varepsilon_t\|_{L^1(0;L)}, \|\tilde{q}^\varepsilon_{tt}\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma}, \tag{2.31}$$

$$\|\tilde{q}^\varepsilon_z\|_{L^\infty}, \|\tilde{q}^\varepsilon_{zt}\|_{L^\infty} \leqslant C\varepsilon^{\gamma}, \|\tilde{q}^\varepsilon_z\|_{L^1(0;L)}, \|\tilde{q}^\varepsilon_{zt}\|_{L^1(0;L)} \leqslant C\varepsilon^{2\gamma}, \tag{2.32}$$

$$\|\tilde{q}^\varepsilon_{zz}\|_{L^\infty} \leqslant C, \|\tilde{q}^\varepsilon_{zz}\|_{L^1(0;L)} \leqslant C\varepsilon^{\gamma}. \tag{2.33}$$

Proof. Using the fact that f is smooth and u_i is piecewise smooth, with discontinuities only at $x = X_j(t)$, we have the following estimates for $\tilde{q}^\varepsilon_1$:

$$\|\tilde{q}^\varepsilon_1\|_{L^\infty} \leqslant C\varepsilon^3$$

where C is a constant which does not depend on ε. Integrating this inequality, we get:

$$\|\tilde{q}^\varepsilon_1\|_{L^1(0;L)} \leqslant C\varepsilon^3.$$

Moreover, if ε is small enough (such that $2\varepsilon^\gamma < r$),

$$\tilde{q}^\varepsilon_{1t}(z,t) = (1-\tilde{\mu}^j)[\dots]_t + \varepsilon^{1-\gamma}\delta^j_t\mu'\left(\frac{\varphi(z,t)-X_j(t)}{\varepsilon^\gamma}\right)[\dots]$$

where the $[\dots]$ is dominated by ε^3. Therefore we have

$$\|\tilde{q}^\varepsilon_{1t}\|_{L^\infty} \leqslant C\varepsilon^3, \|\tilde{q}^\varepsilon_{1t}\|_{L^1(0;L)} \leqslant C\varepsilon^3.$$

Similarly, we prove the same estimates for $\tilde{q}^\varepsilon_{1tt}$. Nevertheless we cannot have such estimates for $\tilde{q}^\varepsilon_{1z}$. Indeed, we have

$$\tilde{q}^\varepsilon_{1z}(z,t) = (1-\tilde{\mu}^j)[\dots]_z - \varepsilon^{-\gamma}\mu'\left(\frac{\varphi(z,t)-X_j(t)}{\varepsilon^\gamma}\right)[\dots]$$

so we just have

$$\|\tilde{q}^{\varepsilon}_{1z}\|_{L^\infty} \leqslant C\varepsilon^{3-\gamma}, \|\tilde{q}^{\varepsilon}_{1z}\|_{L^1(0;L)} \leqslant C\varepsilon^3.$$

Similarly, we prove

$$\|\tilde{q}^{\varepsilon}_{1zt}\|_{L^\infty} \leqslant C\varepsilon^{3-\gamma}, \|\tilde{q}^{\varepsilon}_{1zt}\|_{L^1(0;L)} \leqslant C\varepsilon^3,$$

$$\|\tilde{q}^{\varepsilon}_{1zz}\|_{L^\infty} \leqslant C\varepsilon^{3-2\gamma}, \|\tilde{q}^{\varepsilon}_{1zz}\|_{L^1(0;L)} \leqslant C\varepsilon^{3-\gamma}.$$

Then, we compute estimates for $\tilde{q}^{\varepsilon}_2$:

$$\|\tilde{q}^{\varepsilon}_2\|_{L^\infty}, \|\tilde{q}^{\varepsilon}_{2t}\|_{L^\infty}, \|\tilde{q}^{\varepsilon}_{2tt}\|_{L^\infty} \leqslant C\varepsilon^2,$$

$$\|\tilde{q}^{\varepsilon}_2\|_{L^1(0;L)}, \|\tilde{q}^{\varepsilon}_{2t}\|_{L^1(0;L)}, \|\tilde{q}^{\varepsilon}_{2tt}\|_{L^1(0;L)} \leqslant C\varepsilon^{2+\gamma},$$

$$\|\tilde{q}^{\varepsilon}_{2z}\|_{L^\infty}, \|\tilde{q}^{\varepsilon}_{2zt}\|_{L^\infty} \leqslant C\varepsilon^{2-\gamma}, \|\tilde{q}^{\varepsilon}_{2z}\|_{L^1(0;L)}, \|\tilde{q}^{\varepsilon}_{2zt}\|_{L^1(0;L)} \leqslant C\varepsilon^2.$$

$$\|\tilde{q}^{\varepsilon}_{2zz}\|_{L^\infty} \leqslant C\varepsilon^{2-2\gamma}, \|\tilde{q}^{\varepsilon}_{2zz}\|_{L^1(0;L)} \leqslant C\varepsilon^{2-2\gamma}.$$

Eventually, we use matching conditions (2.9)-(2.10)-(2.11) to prove the estimates on $\tilde{q}^{\varepsilon}_3$. Indeed, the properties of viscous shock profiles provide that terms $o(1)$ can be replaced by $e^{-\alpha|\xi|}$ in the matching conditions with α a positive number. So, we have for $z-(j-1)\frac{L}{m} < r$:

$$\begin{aligned} I^{j\varepsilon}(\varphi(z,t),t) &= V^j\left(\tfrac{z-(j-1)\frac{L}{m}}{\varepsilon},t\right) + \varepsilon V^j_1\left(\tfrac{z-(j-1)\frac{L}{m}}{\varepsilon},t\right) + \varepsilon^2 V^j_2\left(\tfrac{z-(j-1)\frac{L}{m}}{\varepsilon},t\right) \\ &= u^{j\pm}(t) + \varepsilon\left[u^{j\pm}_1(t) + u^{j\pm}_x(t)\left(\tfrac{z-(j-1)\frac{L}{m}-\varepsilon\delta^j_0}{\varepsilon}\right)\right] + \varepsilon^2\Bigg[u^{j\pm}_2(t) \\ &\quad + u^{j\pm}_{1x}(t)\left(\tfrac{z-(j-1)\frac{L}{m}-\varepsilon\delta^j_0}{\varepsilon}\right) + \tfrac{1}{2}u^{j\pm}_{xx}(t)\left(\tfrac{z-(j-1)\frac{L}{m}-\varepsilon\delta^j_0}{\varepsilon}\right)^2 - u^{j\pm}_x(t)\delta^j_1(t)\Bigg] \\ &\quad + \mathcal{O}\left(e^{-\alpha\frac{|z-(j-1)\frac{L}{m}|}{\varepsilon}}\right) \end{aligned}$$

and, using Taylor expansion for $u_i(X_j + (z+(j-1)\frac{L}{m} - \varepsilon\delta^j_0))$,

$$\begin{aligned} O^{\varepsilon}(\varphi(z,t),t) &= u(z+X_j-(j-1)\tfrac{L}{m}-\varepsilon\delta^j,t) \\ &\quad +\varepsilon u_1(z+X_j-(j-1)\tfrac{L}{m}-\varepsilon\delta^j,t) \\ &\quad +\varepsilon^2 u_2(z+X_j-(j-1)\tfrac{L}{m}-\varepsilon\delta^j,t) \\ &= +u^{j\pm}(t) + u^{j\pm}_x(t)(z-(j-1)\tfrac{L}{m}+\varepsilon\delta^j_0+\varepsilon^2\delta^j_1) \\ &\quad +\tfrac{1}{2}u^{j\pm}_{xx}(t)(z-(j-1)\tfrac{L}{m}-\varepsilon\delta^j_0)^2 + \varepsilon u^{j\pm}_1 \\ &\quad +\varepsilon u^{j\pm}_{1x}(z-(j-1)\tfrac{L}{m}-\varepsilon\delta^j_0) + \varepsilon^2 u^{j\pm}_2 \\ &\quad +\mathcal{O}(\varepsilon^3 + (z-(j-1)\tfrac{L}{m}-\varepsilon\delta^j_0)^3). \end{aligned}$$

Since

$$\operatorname{supp}(\tilde{q}_3^\varepsilon) \subset \left\{ (z,t) : \varepsilon^\gamma \leqslant \left| z - (j-1)\frac{L}{m} - \varepsilon\delta^j \right| \leqslant 2\varepsilon^\gamma \right\},$$

we have

$$\mathcal{O}\left(\varepsilon^3 + \left(z - (j-1)\frac{L}{m} - \varepsilon\delta_0^j \right)^3 \right) = \mathcal{O}(\varepsilon^{3\gamma})$$

and we obtain the estimates:

$$\|\tilde{q}_3^\varepsilon\|_{L^\infty}, \|\tilde{q}_{3t}^\varepsilon\|_{L^\infty}, \|\tilde{q}_{3tt}^\varepsilon\|_{L^\infty} \leqslant C\varepsilon^{2\gamma},$$

$$\|\tilde{q}_3^\varepsilon\|_{L^1(0;L)}, \|\tilde{q}_{3t}^\varepsilon\|_{L^1(0;L)}, \|\tilde{q}_{3tt}^\varepsilon\|_{L^1(0;L)} \leqslant C\varepsilon^{3\gamma},$$

$$\|\tilde{q}_{3z}^\varepsilon\|_{L^\infty}, \|\tilde{q}_{3zt}^\varepsilon\|_{L^\infty} \leqslant C\varepsilon^{\gamma}, \|\tilde{q}_{3z}^\varepsilon\|_{L^1(0;L)}, \|\tilde{q}_{3zt}^\varepsilon\|_{L^1(0;L)} \leqslant C\varepsilon^{2\gamma},$$

$$\|\tilde{q}_{3zz}^\varepsilon\|_{L^\infty} \leqslant C, \|\tilde{q}_{3zz}^\varepsilon\|_{L^1(0;L)} \leqslant C\varepsilon^{\gamma}.$$

This ends the proof of the lemma. □

We now end the proof of Theorem 2.2 with estimates on $\tilde{R}^\varepsilon$. Since d^ε verifies equation (2.29) with zero initial data, $\tilde{d}^\varepsilon$ is given by

$$\tilde{d}^\varepsilon = -\int_0^t \int_{\mathbb{R}} k^\varepsilon(z,y,t,\tau)\tilde{q}^\varepsilon(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$

where the kernel k^ε is defined by

$$k^\varepsilon(z,y,t,\tau) = \frac{-e^{\kappa(t-\tau)}\varphi_z(y,\tau)}{\sqrt{4\pi\varepsilon(t-\tau)}} \exp\left(-\frac{(\varphi(z,t)-\varphi(y,\tau))^2}{4\varepsilon(t-\tau)} \right).$$

We easily verify that $\|k^\varepsilon(\cdot,y,\cdot,\tau)\|_{L^1} + \sqrt{\varepsilon}\|k_z^\varepsilon(\cdot,y,\cdot,\tau)\|_{L^1}$ is independent of ε and we deduce the estimates on $\tilde{d}^\varepsilon$

$$\|\tilde{d}^\varepsilon\|_{L^1} \leqslant C\varepsilon^{3\gamma}, \|\tilde{d}^\varepsilon\|_{L^\infty} \leqslant C\varepsilon^{2\gamma},$$
$$\|\tilde{d}_z^\varepsilon\|_{L^1} \leqslant C\varepsilon^{3\gamma-1/2}, \|\tilde{d}_z^\varepsilon\|_{L^\infty} \leqslant C\varepsilon^{2\gamma-1/2}.$$

Then, we derive (2.29) to obtain the equation on $\tilde{d}_t^\varepsilon$

$$\begin{aligned}\tilde{d}_{tt}^\varepsilon + \left(\varepsilon\frac{\varphi_{zz}}{\varphi_z^3} - \frac{\varphi_t}{\varphi_z}\right)\tilde{d}_{tz}^\varepsilon - \frac{\varepsilon}{\varphi_z^2}\tilde{d}_{tzz}^\varepsilon - \left(\kappa - 2\frac{\varphi_{zt}}{\varphi_z}\right)\tilde{d}_t^\varepsilon \\ = -\tilde{q}_t^\varepsilon - \left(\varepsilon\frac{\varphi_{zz}}{\varphi_z^3} - \frac{\varphi_t}{\varphi_z}\right)_t \tilde{d}_z^\varepsilon - \frac{2\varphi_{zt}}{\varphi_z}\left(\left(\varepsilon\frac{\varphi_{zz}}{\varphi_z^3} - \frac{\varphi_t}{\varphi_z}\right)\tilde{d}_z^\varepsilon - \kappa\tilde{d}^\varepsilon + \tilde{q}^\varepsilon\right)\end{aligned}$$

with initial data

$$\tilde{d}_t^\varepsilon(z,0) = -\tilde{q}^\varepsilon(z,0).$$

We deduce the estimates

$$\|\tilde{d}^\varepsilon_t\|_{L^1} \leqslant C\varepsilon^{3\gamma-1/2}, \|\tilde{d}^\varepsilon_t\|_{L^\infty} \leqslant C\varepsilon^{2\gamma-1/2},$$
$$\|\tilde{d}^\varepsilon_{tz}\|_{L^1} \leqslant C\varepsilon^{3\gamma-1}, \|\tilde{d}^\varepsilon_{tz}\|_{L^\infty} \leqslant C\varepsilon^{2\gamma-1}.$$

Taking again the time derivative of the equation we obtain estimates on $\tilde{d}^\varepsilon_{tt}$

$$\|\tilde{d}^\varepsilon_{tt}\|_{L^1} \leqslant C\varepsilon^{3\gamma-1}, \|\tilde{d}^\varepsilon_{tt}\|_{L^\infty} \leqslant C\varepsilon^{2\gamma-1}.$$

Using space derivative, we obtain estimate on $\tilde{d}^\varepsilon_{zz}$

$$\|\tilde{d}^\varepsilon_{zz}\|_{L^1} \leqslant C\varepsilon^{2\gamma-1/2}, \|\tilde{d}^\varepsilon_{tt}\|_{L^\infty} \leqslant C\varepsilon^{\gamma-1/2}.$$

We recall that $\tilde{R}^\varepsilon = f(\tilde{u}^\varepsilon_{app}) - f(\tilde{u}^\varepsilon_{app} - \tilde{d}^\varepsilon)$ and using L^∞-estimates on $\tilde{u}^\varepsilon_{app}$

$$\|\tilde{u}^\varepsilon_{app}\|_{L^\infty} + \|(\tilde{u}^\varepsilon_{app})_t\|_{L^\infty} + \|(\tilde{u}^\varepsilon_{app})_{tt}\|_{L^\infty} \leqslant C,$$
$$\varepsilon\|(\tilde{u}^\varepsilon_{app})_z\|_{L^\infty} + \varepsilon\|(\tilde{u}^\varepsilon_{app})_{tz}\|_{L^\infty} + \varepsilon^2\|(\tilde{u}^\varepsilon_{app})_{zz}\|_{L^\infty} \leqslant C,$$

we finally obtain the estimates on $\tilde{R}^\varepsilon$. This ends the proof of Theorem 2.2.

2.3 Estimates on the Green's function

We now consider the linear operator

$$L^\varepsilon \tilde{w} = \tilde{w}_t + \frac{1}{\varphi_z}\left(\mathrm{d}f(\tilde{u}^\varepsilon_{app}) - \varphi_t + \varepsilon\frac{\varphi_{zz}}{\varphi_z^2}\right)\cdot \tilde{w}_z - \varepsilon\frac{1}{\varphi_z^2}\tilde{w}_{zz} - \kappa\tilde{w}.$$

The aim of this section is to prove the following theorem:

Theorem 2.4. *There exists a Green's function $G^\varepsilon(t,\tau,z,y)$ of the linear operator L^ε defined for $0 \leqslant \tau,\ t \leqslant T^*, z, y \in \mathbb{R}$ such that $G^\varepsilon(t,\tau,z,y) = 0$ if $\tau > t$ and*

$$\sup_{y,\tau\leqslant T^*}\int_0^{T^*}\int_{\mathbb{R}} |G^\varepsilon(t,\tau,z,y)|\,\mathrm{d}z\,\mathrm{d}t + \sqrt{\varepsilon}\sup_{y,\tau\leqslant T^*}\int_0^{T^*}\int_{\mathbb{R}} |\partial_z G^\varepsilon(t,\tau,z,y)|\,\mathrm{d}z\,\mathrm{d}t \leqslant C \tag{2.34}$$

where C is positive and does not depend on ε.

Remark 2.2. If we replace κ by a bounded smooth function h with bound independent of ε, estimate (2.34) still holds. We will prove this theorem with this general assumption.

To find estimates on this Green's function, we use approximations of the Green's function both near the shock and far away from the shocks.

First, we recall the method of iterative construction of the Green's function of E. Grenier and F. Rousset [34]. Then, to approximate the Green's function near the shocks, we recall the result of K. Zumbrun and P. Howard [74] about Green's function for pure viscous profile problem. Far away from the shocks, we use characteristic curves to build some approximate Green's functions. Finally, we combine all these Green's functions to obtain an approximate Green's function of L^ε and we find bounds on the error terms.

2.3.1 Method

Here, we recall the method used by E. Grenier and F. Rousset in [34]. We want to construct an approximate Green's function G^ε_{app} of L^ε in the form

$$G^\varepsilon_{app}(t,\tau,z,y) = \sum_{k=1}^{N} S_k(t,\tau,z,y)\Pi_k(\tau,y),$$

where S_k are Green's kernels which satisfy (2.34) and $\Pi_k \in \mathcal{C}^\infty([0,T^*]\times\mathbb{R},\mathcal{L}(\mathbb{R}^n))$ are such that

$$\|\Pi_k(t,x)v\| \leqslant C\|v\|, \quad \forall x \geqslant 0, t\in[0;T^*], v\in\mathbb{R}^n$$

and

$$\sum_{k=1}^{N} \Pi_k = \mathrm{Id}\,.$$

We next define the error $R_k(\cdot,\tau,\cdot,y) = L^\varepsilon S_k(\tau,y)$ for $k=1,\dots,N$, and the matrix of errors: $\mathcal{M}(T_1,T_2) = (\sigma_{kl}(T_1,T_2))_{1\leqslant k,l\leqslant N}$ with

$$\sigma_{kl}(T_1,T_2) = \sup_{T_1\leqslant\tau\leqslant T_2, y\in\operatorname{supp}\Pi_l} \int_{T_1}^{T_2}\int_{\mathbb{R}} |\Pi_k(t,z)R_l(t,\tau,z,y)|\,\mathrm{d}z\,\mathrm{d}t.$$

Thanks to Theorem 2.2 of [34], we just have to prove that there exists $\varepsilon_2 > 0$ such that $0 < T_2 - T_1 < \varepsilon_2$ implies

$$\lim_{p\to\infty} \mathcal{M}^p(T_1,T_2) = 0.$$

Remark 2.3. Since we consider the error in a matrix, we need to consider a finite number of Green's kernel S_k. We will see in the following that the number of these kernels is proportional to the number of shocks per period. So, this method does not allow us to treat the case of a non-periodic perturbation.

2.3.2 Near the shocks

Near the shock j, we can approximate

$$\frac{1}{\varphi_z}\left(\mathrm{d}f(\tilde{u}^\varepsilon_{app}) - \varphi_t + \varepsilon\frac{\varphi_{zz}}{\varphi_z^2}\right)$$

by

$$\mathrm{d}f\left(V^j\left(\frac{z-(j-1)\frac{L}{m}}{\varepsilon},\tau\right)\right) - X_j'$$

and we forget zeroth order term. Therefore, we search the Green's functions for the linear operators

$$L_\tau^{\varepsilon j} w = \partial_t w + \left(\mathrm{d}f\left(V^j\left(\frac{z-(j-1)\frac{L}{m}}{\varepsilon},\tau\right)\right) - X_j'(\tau)\right) w_z - \varepsilon w_{zz}$$

which depend on j and $\tau < T^*$. As in [63], we remark that these Green's functions $G_\tau^{Sj}(t,z,y)$ verify

$$G_\tau^{Sj}(t,z,y) = \frac{1}{\varepsilon} G_\tau^{HZj}\left(\frac{t}{\varepsilon}, \frac{z-(j-1)\frac{L}{m}}{\varepsilon}, \frac{y-(j-1)\frac{L}{m}}{\varepsilon}\right)$$

where G_τ^{HZj} is the Green's function related to the operator

$$L_\tau^j w = \partial_t w + (\mathrm{d}f(V^j(z,\tau)) - X_j'(\tau)) w_z - w_{zz}.$$

In [74], K. Zumbrun and P. Howard obtained estimates on the Green's functions which will be useful to obtain estimates for our operators. Let us denote by $\tilde{a}_i^{j\pm}(\tau)$, and $r_i^{j\pm}(\tau)$ the eigenvalues and the associated eigenvectors of $\mathrm{d}f(u^{j\pm}(\tau)) - X_j'(\tau)$.

Proposition 2.2. *Under hypothesis* **(H3)**, *we have*

$$\begin{aligned} G_\tau^{Sj}\left(t, z+(j-1)\frac{L}{m}, y+(j-1)\frac{L}{m}\right) = &\sum_{i,\tilde{a}_i^{j+}(\tau)>0} \mathcal{O}\left(\frac{\exp\left(-\frac{(z-\tilde{a}_i^{j+}(\tau)t)^2}{M\varepsilon t}\right)}{\sqrt{\varepsilon t}}\right) r_i^{j+}(\tau)\chi_{z\geqslant 0} \\ &+ \sum_{i,\tilde{a}_i^{j-}(\tau)<0} \mathcal{O}\left(\frac{\exp\left(-\frac{(z-\tilde{a}_i^{j-}(\tau)t)^2}{M\varepsilon t}\right)}{\sqrt{\varepsilon t}}\right) r_i^{j-}(\tau)\chi_{z\leqslant 0} \\ &+ \mathcal{O}\left(\frac{\exp\left(-\frac{(z-y)^2}{M\varepsilon t}\right)}{\sqrt{\varepsilon t}} e^{-\sigma\frac{t}{\varepsilon}}\right) \end{aligned} \tag{2.35}$$

$$
\begin{aligned}
\partial_z G^{Sj}_\tau\left(t, z+(j-1)\frac{L}{m}, y+(j-1)\frac{L}{m}\right) = &\sum_{i,\tilde{a}_i^{j+}(\tau)>0} \mathcal{O}\left(\frac{\exp\left(-\frac{(z-\tilde{a}_i^{j+}(\tau)t)^2}{M\varepsilon t}\right)}{\varepsilon t}\right) r_i^{j+}(\tau)\chi_{z\geqslant 0}\\
&+\sum_{i,\tilde{a}_i^{j-}(\tau)<0} \mathcal{O}\left(\frac{\exp\left(-\frac{(z-\tilde{a}_i^{j-}(\tau)t)^2}{M\varepsilon t}\right)}{\varepsilon t}\right) r_i^{j-}(\tau)\chi_{z\leqslant 0}\\
&+\mathcal{O}\left(\frac{\exp\left(-\frac{(z-y)^2}{M\varepsilon t}\right)}{\varepsilon t} e^{-\sigma\frac{t}{\varepsilon}}\right),
\end{aligned}
\tag{2.36}
$$

where M and σ are positive constants, and χ designs characteristic function. Moreover, $\mathcal{O}$'s are at least linear forms, locally bounded in y and uniformly bounded in (t,z).

2.3.3 Far away from the shocks

As in the previous section, we do not search Green's function for L^ε but for the approximate operator $\tilde{L}^\varepsilon$ defined by

$$
\tilde{L}^\varepsilon w = w_t + \frac{1}{\varphi_z}\left(\mathrm{d}f(u(\varphi(z,t),t)) - \varphi_t + \varepsilon\frac{\varphi_{zz}}{\varphi_z^2}\right)\cdot w_z - \frac{\varepsilon}{\varphi_z^2} w_{zz}.
$$

We remark that, as in the previous section, we forget the terms in w.

Recall that $\mathrm{d}f(u(x,t))$ is diagonalizable for all x, t, so we can write

$$
\mathrm{d}f(u(\varphi(z,t),t)) = P(u(\varphi(z,t),t))D(u(\varphi(z,t),t))P(u(\varphi(z,t),t))^{-1}
$$

with $D(u(\varphi(z,t),t)) = \mathrm{diag}(\lambda_i(u(\varphi(z,t),t)))$.

To obtain approximation of the Green's function between two shocks, we define j approximate problems on $\mathbb{R}$, with continuous solutions. First, we set

$$
\lambda_i^j(\varphi(z,t),t) = \begin{cases} \lambda_i(u(X_j^+(t),t)) & \text{if } z \in \left]-\infty; (j-1)\frac{L}{m}+\varepsilon\delta^j\right], \\ \lambda_i(u(\varphi(z,t),t)) & \text{if } z \in \left](j-1)\frac{L}{m}+\varepsilon\delta^j; j\frac{L}{m}+\varepsilon\delta^{j+1}\right[, \\ \lambda_i(u(X_{j+1}^-(t),t)) & \text{if } z \in \left[j\frac{L}{m}+\varepsilon\delta^{j+1}; +\infty\right[. \end{cases}
$$

Then, we want to find approximate Green's functions for the scalar operators

$$
L_i^j w = w_t + \frac{1}{\varphi_z}\left(\lambda_i^j - \varphi_t + \varepsilon\frac{\varphi_{zz}}{\varphi_z^2}\right) w_z - \frac{\varepsilon}{\varphi_z^2} w_{zz}.
$$

To do so, we define characteristic curves $\chi_i^j(t,\tau,y)$ by

$$\begin{cases} \partial_t \chi_i^j(t,\tau,y) = \lambda_i^j(\chi_i^j(t,\tau,y),t), & t \geqslant \tau, \\ \chi_i^j(\tau,\tau,y) = y, \end{cases}$$

and the approximate Green's functions

$$G_i^j(t,\tau,z,y) = \frac{\varphi_z(y,\tau)}{\sqrt{4\pi\varepsilon(t-\tau)}} \exp\left(-\frac{(\varphi(z,t) - \chi_i^j(t,\tau,\varphi(y,\tau)))^2}{4\varepsilon(t-\tau)} \right).$$

We easily compute the error committed here

$$L_i^j G_i^j = (\lambda_i^j(\varphi(z,t),t) - \lambda_i^j(\chi_i^j(t,\tau,\varphi(y,\tau)),t)) G_{iz}^j(t,\tau,z,y).$$

Before we build the whole Green's function, we introduce some notations. First, we write

$$G^j = \mathrm{diag}(G_i^j).$$

In the sequel, we need to distinguish at each shock the outgoing waves to the incoming waves. We define

$$\begin{array}{rcll} D^{-in} & = & \mathrm{diag}(0,\dots,0,1,\dots,1), & \text{with } k-1 \text{ unit coefficients}, \\ D^{-out} & = & \mathrm{diag}(1,\dots,1,0,\dots,0), & \text{with } k-1 \text{ null coefficients}, \\ D^{+in} & = & \mathrm{diag}(1,\dots,1,0,\dots,0), & \text{with } k \text{ unit coefficients}, \\ D^{+out} & = & \mathrm{diag}(0,\dots,0,1,\dots,1), & \text{with } k \text{ null coefficients}, \end{array}$$

so that $D^{\pm in} + D^{\pm out} = \mathrm{Id}$.

Finally, we define the projections

$$\mathcal{P}^{\pm in}(t,z) = P(t,z) D^{\pm in} P(t,z)^{-1},$$
$$\mathcal{P}^{\pm out}(t,z) = P(t,z) D^{\pm out} P(t,z)^{-1}.$$

2.3.4 Approximate Green's function

Since the shocks are non-characteristic Lax shocks, we have the following inequality on a neighbourhood of each shock $j = 1,\dots,m$:

$$|\lambda_i(\tilde{u}(z,t)) - X_j(t)| > C > 0 \text{ if } (j-1)\frac{L}{m} - 4\eta < z < (j-1)\frac{L}{m} + 4\eta, i = 1,\dots,n.$$

We can assume that η is such that $4\eta < r$ so that $\alpha_j \equiv 1$ in (2.27) on the previous neighbourhood.

Furthermore, we need some cut-off smooth functions

$$K^+(z) = \begin{cases} 0 \text{ if } z \leqslant 1, \\ 1 \text{ if } z \geqslant 2 \end{cases} \quad \text{and } K^-(z) = \begin{cases} 1 \text{ if } z \leqslant -2, \\ 0 \text{ if } z \geqslant -1. \end{cases}$$

We also assume the cut-off function μ already used to read as $\mu = (1-K^+)(1-K^-)$. We can now build an approximate Green's function in the form

$$G^\varepsilon_{app}(t,\tau,z,y) = \sum_{j=1}^{m} \sum_{k=0}^{7} S^j_k(t,\tau,z,y)\Pi^j_k(\tau,y)$$

where the Green's kernels are periodic with period $(0,0,L,L)$:

$$S^j_k(t,\tau,z,y) = \sum_{l\in\mathbb{Z}} \tilde{S}^j_k(t,\tau,z+lL,y+lL)$$

with

$$\begin{aligned}
\tilde{S}^j_0(t,\tau,z,y) &= \mu\left(\frac{z-(j-1)\frac{L}{m}}{2\eta}\right)\mu\left(\frac{z-(j-1)\frac{L}{m}}{M_3\varepsilon}\right)G^{Sj}_\tau(t-\tau,z,y),\\
\tilde{S}^j_{1,2}(t,\tau,z,y) &= \mu\left(\frac{\cdots}{2\eta}\right)K^+\left(\frac{z-(j-1)\frac{L}{m}}{M_1\varepsilon}\right)P(t,z)D^{+out}G^j(t,\tau,z,y)P(\tau,y)^{-1},\\
\tilde{S}^j_3(t,\tau,z,y) &= \mu\left(\frac{\cdots}{2\eta}\right)K^+\left(\frac{z-(j-1)\frac{L}{m}}{M_1\varepsilon}\right)P(t,z)D^{+in}G^j(t,\tau,z,y)P(\tau,y)^{-1},\\
\tilde{S}^j_4(t,\tau,z,y) &= \left(K^+\left(\frac{4(z-(j-1)\frac{L}{m})}{\eta}\right)+K^-\left(\frac{4(z-j\frac{L}{m})}{\eta}\right)-1\right)P(t,z)G^jP(\tau,y)^{-1},\\
\tilde{S}^j_5(t,\tau,z,y) &= \mu\left(\frac{z-j\frac{L}{m}}{2\eta}\right)K^-\left(\frac{z-j\frac{L}{m}}{M_1\varepsilon}\right)P(t,z)D^{-in}G^j(t,\tau,z,y)P(\tau,y)^{-1},\\
\tilde{S}^j_{6,7}(t,\tau,z,y) &= \mu\left(\frac{\cdots}{2\eta}\right)K^-\left(\frac{z-j\frac{L}{m}}{M_1\varepsilon}\right)P(t,z)D^{-out}G^j(t,\tau,z,y)P(\tau,y)^{-1},
\end{aligned}$$

and the projectors are also periodic:

$$\Pi^j_k(\tau,y) = \sum_{l\in\mathbb{Z}} \tilde{\Pi}^j_k(\tau,y+lL)$$

with

$$
\begin{aligned}
\tilde{\Pi}_0^j(\tau,y) &= \mu\left(\frac{y-(j-1)\frac{L}{m}}{\eta}\right)\mu\left(\frac{y-(j-1)\frac{L}{m}}{M_2\varepsilon}\right),\\
\tilde{\Pi}_1^j(\tau,y) &= \mu\left(\frac{\cdots}{\eta}\right)K^+\left(\frac{y-(j-1)\frac{L}{m}}{M_2\varepsilon}\right)\left(1-K^+\left(\frac{2(y-(j-1)\frac{L}{m})}{M_3\varepsilon}\right)\right)\mathcal{P}^{+out}(\tau,y),\\
\tilde{\Pi}_2^j(\tau,y) &= \mu\left(\frac{\cdots}{\eta}\right)K^+\left(\frac{2(y-(j-1)\frac{L}{m})}{M_3\varepsilon}\right)\mathcal{P}^{+out}(\tau,y),\\
\tilde{\Pi}_3^j(\tau,y) &= \mu\left(\frac{\cdots}{\eta}\right)K^+\left(\frac{y-(j-1)\frac{L}{m}}{M_2\varepsilon}\right)\mathcal{P}^{+in}(\tau,y),\\
\tilde{\Pi}_4^j(\tau,y) &= K^+\left(\frac{y-(j-1)\frac{L}{m}}{\eta}\right)+K^-\left(\frac{y-j\frac{L}{m}}{\eta}\right)-1,\\
\tilde{\Pi}_5^j(\tau,y) &= \mu\left(\frac{y-j\frac{L}{m}}{\eta}\right)K^-\left(\frac{y-j\frac{L}{m}}{M_2\varepsilon}\right)\mathcal{P}^{-in}(\tau,y),\\
\tilde{\Pi}_6^j(\tau,y) &= \mu\left(\frac{\cdots}{\eta}\right)K^-\left(\frac{2(y-j\frac{L}{m})}{M_3\varepsilon}\right)\mathcal{P}^{-out}(\tau,y),\\
\tilde{\Pi}_7^j(\tau,y) &= \mu\left(\frac{\cdots}{\eta}\right)K^-\left(\frac{y-j\frac{L}{m}}{M_2\varepsilon}\right)\left(1-K^-\left(\frac{2(y-j\frac{L}{m})}{M_3\varepsilon}\right)\right)\mathcal{P}^{-out}(\tau,y)).
\end{aligned}
$$

It appears that all the Green's kernels can be written in the following form:

$$S_k^j(t,\tau,z,y) = T(z)S(t,\tau,z,y) \tag{2.37}$$

where T is a truncation function.

We will choose the three constants M_1, M_2, M_3 at the end of the estimates on the error matrix so that they verify

$$4M_1 \leqslant M_2 \leqslant \frac{1}{4}M_3.$$

These inequalities are necessary to have

$$\sum_{j,k}\Pi_k^j \equiv 1$$

and

$$G^{app}(\tau,\tau,z,y) = \delta_y(z)\,\mathrm{Id}\,.$$

Under these notations, $\tilde{S}_0^j$ describes the viscous dynamic at the shock j, $\tilde{S}_1^j$ the creation of outgoing waves in a vicinity at the right of the shock j, $\tilde{S}_2^j$ the creation and propagation of outgoing waves away from the shock j, at its right, $\tilde{S}_3^j$ the creation and propagation of incoming waves at the right of the shock j. $\tilde{S}_4^j$ describes the propagation of the waves between the shocks j and $j+1$ Moreover, the kernels $\tilde{S}_7^j, \tilde{S}_6^j, \tilde{S}_5^j$ are the symmetric of respectively $\tilde{S}_1^j, \tilde{S}_2^j, \tilde{S}_3^j$ for the left of the shock $j+1$. We summarize this splitting in Figure 2.4.

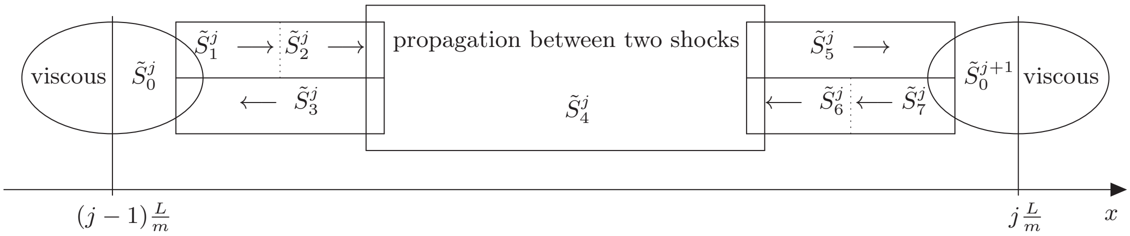


Figure 2.4: Summarize of the splitting by Green's kernels.

2.3.5 Bounds on the error matrix

As said in Subsection 2.3.1, to prove Theorem 2.4, it remains to prove that $\mathcal{M}^p$ converges to 0 when p goes to ∞. Since the coefficients of $\mathcal{M}$ are non-negative, it suffices to prove that $\mathcal{M}$ is bounded above by an other matrix which has the "good" convergence.

To bound the error terms, we use the same method than F. Rousset in [63]. We split all the error terms into two parts: the truncation of the error on the kernel and the commutator:

$$R_k^j = E_{k1}^j + E_{k2}^j$$

where, with the notation of (2.37),

$$E_{k1}^j(t,\tau,z,y) = T(z)L^\varepsilon S(t,\tau,z,y) \text{ and } E_{2k}^j(t,\tau,z,y) = [L^\varepsilon, T(z)]S(t,\tau,z,y).$$

Lemma 2.3. *We have the estimates*

at shock j: $\tilde{R}_0^j$

$\|\mathbf{1}_{|y-(j-1)L/m|\leqslant 2M_2\varepsilon} E_{01}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_1(T+\varepsilon),$

$\|\mathbf{1}_{|y-(j-1)L/m|\leqslant 2M_2\varepsilon}\mathbf{1}_{\pm z\geqslant 0}\mathcal{P}^\pm_{out}E_{02}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_2,$

$\|\mathbf{1}_{|y-(j-1)L/m|\leqslant 2M_2\varepsilon}\mathbf{1}_{\pm z\geqslant 0}\mathcal{P}^\pm_{in}E_{02}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_3 + C_2 T;$

for the outgoing waves: $\tilde{R}_1^j, \tilde{R}_2^j, \tilde{R}_6^j, \tilde{R}_7^j$ *. Let $M \geqslant M_2$. we have:*

$$\|\mathbf{1}_{y-(j-1)L/m \geqslant M\varepsilon} E_{11}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}}, \|\mathbf{1}_{y \geqslant M\varepsilon} E_{21}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_4(T+\varepsilon^{2\gamma-1}) + C_5,$$

$$\|\mathbf{1}_{y-(j-1)L/m \leqslant -M\varepsilon} E_{61}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}}, \|\mathbf{1}_{y \leqslant -M\varepsilon} E_{71}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_4(T+\varepsilon^{2\gamma-1}) + C_5,$$

$$\|\mathbf{1}_{y-(j-1)L/m \geqslant M\varepsilon} E_{12}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}}, \|\mathbf{1}_{y \geqslant M\varepsilon} E_{22}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_5 + C_1(T+\varepsilon),$$

$$\|\mathbf{1}_{y-(j-1)L/m \leqslant -M\varepsilon} E_{62}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}}, \|\mathbf{1}_{y \leqslant -M\varepsilon} E_{72}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_5 + C_1(T+\varepsilon);$$

for the incoming waves: $\tilde{R}_3^j, \tilde{R}_5^j$

$$\|\mathbf{1}_{y-(j-1)L/m \geqslant M\varepsilon} E_{31}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}}, \|\mathbf{1}_{y \leqslant -M\varepsilon} E_{51}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_6(T+\varepsilon^{2\gamma-1}) + C_7,$$

$$\|\mathbf{1}_{y-(j-1)L/m \geqslant M\varepsilon} E_{32}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}}, \|\mathbf{1}_{y \leqslant -M\varepsilon} E_{52}^j(t,\tau,z,y)\|_{L^\infty_{\tau,y},L^1_{t,z}} \leqslant C_8 + C_1(T+\varepsilon);$$

between two shocks: $\tilde{R}_4^j$

$$\|R_4^j\|_{L^\infty_{\tau,y},L^1_{t,z}}, \leqslant C_8(T+\varepsilon),$$

where C_1 is locally bounded in M_2, M_3,C_2 is locally bounded in M_2 uniformly in M_3, C_3 depends only on M_2 and M_3 and goes to 0 as $M_3 \to +\infty$, C_4 is independent of M_1, M_2 and M_3, C_5 depends only on M and goes to 0 as $M \to +\infty$, C_6 is locally bounded in M_1, C_7 goes to 0 as $M_1 \to +\infty$, and C_8 is bounded uniformly in M_1.

Proof. We do not give here the complete proof of the lemma. Mainly, it deals with terms that are not treated in [63]: zeroth order terms and terms related to φ or η.

First, we consider the error at the shock j. On the support of E_{01}^j, we have that $\varphi_z(\cdot,t) \equiv 1$ and $\varphi_t(\cdot,t) \equiv X_j'(t) - \varepsilon\delta_t^j(t)$ so we obtain

$$E_{01}^j = \mu\left(\frac{z-(j-1)\frac{L}{m}}{2\eta}\right)\mu\left(\frac{z-(j-1)\frac{L}{m}}{M_3\varepsilon}\right)\Bigg[h\; G_\tau^{Sj}(t-\tau,z,y) + \left(\mathrm{d}f(\tilde{u}^\varepsilon_{app}) - \mathrm{d}f\left(V^j\left(\frac{z-(j-1)/m}{\varepsilon},\tau\right)\right) + X_j'(\tau) - X_j'(t) + \varepsilon\delta_t^j\right)G_{\tau z}^{Sj}\Bigg]$$

where h is a bounded function of t,z. Hence, using the fact that $\mathrm{d}f$ and X_j are smooth and the expression of u^ε_{app}, we have

$$|E_{01}^j| \leqslant C\mu\left(\frac{z-(j-1)\frac{L}{m}}{2\eta}\right)\mu\left(\frac{z-(j-1)\frac{L}{m}}{M_3\varepsilon}\right)\left[(|t-\tau|+\varepsilon)|G_{\tau z}^{Sj}| + |G_\tau^{Sj}|\right]$$

where C is locally bounded in M_3. The calculations of [63] give directly a bound for the first term, in $G^{Sj}_{\tau z}$. So, it only remains to bound the integral:

$$\int_\tau^T \int_{(j-1)L/m-2M_3\varepsilon}^{(j-1)L/m+2M_3\varepsilon} |G^{Sj}_\tau(t-\tau,z,y)| \,\mathrm{d}z\,\mathrm{d}t,$$

and, thanks to Proposition 2.2, to estimate:

$$\int_\tau^T \int_{-M_3\varepsilon}^{M_3\varepsilon} \frac{1}{\sqrt{\varepsilon(t-\tau)}} \exp\left(-\frac{(z-a(t-\tau))^2}{M\varepsilon(t-\tau)}\right) \mathrm{d}z\,\mathrm{d}t.$$

Using the classical change of variable $z' = \frac{z-a(t-\tau)}{\sqrt{M\varepsilon(t-\tau)}}$, we obtain the bound

$$\int_\tau^T \int_{(j-1)L/m-2M_3\varepsilon}^{(j-1)L/m+2M_3\varepsilon} |G^{Sj}_\tau(t-\tau,z,y)| \,\mathrm{d}z\,\mathrm{d}t \leqslant CT$$

so this concludes the estimate for E^j_{01}. For E^j_{02}, the linear term in G^{Sj}_τ disappears in the commutator. Hence, there is no change with the proof of F. Rousset.

For the estimates on $\tilde{R}^j_i, i = 1,2,3,5,6,$ and 7, we first remark that $\varphi_z \equiv 1$ on the support of the errors, so E^j_{i1} is bounded as in [63], except for the zeroth order term which is treated as in the case of E^j_{01}. So, we only consider the estimate on E^j_{i2}. We first remark that the support of this error is not of size ε. Indeed, the truncation $\mu\left(\frac{z-(j-1)\frac{L}{m}}{2\eta}\right)$ adds some error terms with support of size 4η. Though, we have to bound:

$$\begin{aligned}\int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} & (\mathrm{d}f(\tilde{u}^\varepsilon_{app}) - X'_j + \varepsilon\delta^j_t)\frac{1}{2\eta}\mu' K^+ P D^{+out} G^j P^{-1} \\ & -\varepsilon\left[\frac{1}{4\eta^2}\mu'' K^+ P D^{+out} G^j P^{-1} + \frac{1}{2\eta}\mu' K^+ (P D^{+out} G^j)_z P^{-1}\right] \mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

The two terms in G^j are bounded by CT, and the term in $(PD^{+out}G^j)_z$ is bounded by $C(\varepsilon+T)$. Indeed, for $1 \leqslant i \leqslant n$,

$$\begin{aligned}&\int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} |G^j_i| \\ &\quad = \int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} \frac{|\varphi_z(y,\tau)|}{\sqrt{4\pi\varepsilon(t-\tau)}} \exp\left(-\frac{(\varphi(z,t)-\chi^j_i(t,\tau,\varphi(y,\tau)))^2}{4\varepsilon(t-\tau)}\right) \mathrm{d}z\,\mathrm{d}t.\end{aligned}$$

Hence, using the change of variable $z' = \frac{\varphi(z,t)-\chi^j_i(t,\tau,\varphi(y,\tau))}{\sqrt{4\varepsilon(t-\tau)}}$, we obtain

$$\int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} |G^j_i| \leqslant C\int_\tau^T \int_{\mathbb{R}} e^{-z'^2}\,\mathrm{d}z'\,\mathrm{d}t \leqslant CT.$$

Similarly, we have

$$\int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} |G_{iz}^j|$$
$$= \int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} \frac{|\varphi_z(y,\tau)|}{\sqrt{4\pi\varepsilon(t-\tau)}} \exp\left(-\frac{(\varphi-\chi_i^j)^2}{4\varepsilon(t-\tau)}\right)\left(-\varphi_z\frac{\varphi-\chi_i^j}{2\varepsilon(t-\tau)}\right) \mathrm{d}z\,\mathrm{d}t.$$

And with the same change of variable, we obtain:

$$\varepsilon\int_\tau^T \int_{(j-1)\frac{L}{m}+M_1\varepsilon}^{(j-1)\frac{L}{m}+4\eta} |G_{iz}^j| \leqslant C\sqrt{\varepsilon}\int_\tau^T \frac{1}{\sqrt{t-\tau}}\int_{\mathbb{R}} |z'|e^{-z'^2}\,\mathrm{d}z'\,\mathrm{d}t \leqslant C(T+\varepsilon),$$

which gives the estimate for E_{i2}^j.

Since there is not new difficulty in the proof of the estimates on R_4^j, we do not develop it here. $\square$

We now use Lemma 2.3 to bound the matrix $\mathcal{M}$. Since the Green's kernel depends on the shock, we note

$$\sigma_{kl}^{ij}(T_1,T_2) = \sup_{T_1\leqslant\tau\leqslant T_2,\, y\in\operatorname{supp}\Pi_l^j} \int_{T_1}^{T_2}\int_{\mathbb{R}} |\Pi_k^i(t,z)R_l^j(t,\tau,z,y)|\,\mathrm{d}z\,\mathrm{d}t.$$

Since two shocks do not interact, the error coefficients σ_{kl}^{ij} vanish for $i\neq j$ and $kl\neq 0$ and for $|i-j|>1$. So the error matrix $\mathcal{M}$ is bounded:

$$\mathcal{M} \leqslant \begin{pmatrix} M_1 & M_2 & & & & M_3 \\ M_3 & M_1 & M_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & M_2 \\ M_2 & & & & M_3 & M_1 \end{pmatrix}$$

where M_2 is null except on the first column, and M_3 is null except on the first line. Moreover, using Lemma 2.3, and it was done in [63], we can choose $\alpha < 1/2$ and

M_1, M_2 such that when $M_3 \to +\infty, \varepsilon, T \to 0$, the matrices tend to

$$M_1 \to \begin{pmatrix} \cdot & \alpha & \cdot & C & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ C & \alpha & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & C & \cdot & C & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \alpha \end{pmatrix}, \quad M_2 \to \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$M_3 \to \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & C & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

So that $\mathcal{M} \to \tilde{\mathcal{M}}$ where the eigenvalues of $\tilde{\mathcal{M}}$ are 0 and 2α. We can now conclude that we have $\mathcal{M}^p \to 0$ when $p \to \infty$. This ends the proof of Theorem 2.4.

2.4 Convergence

The purpose of this section is to prove Theorem 2.3 and to conclude the proof of Theorem 2.1. For this sake, it remains to show that solution of

$$\begin{cases} L^\varepsilon \tilde{w} = -\tilde{R}^\varepsilon - Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z), & (2.38) \\ \tilde{w}(z,0) = 0 & (2.39) \end{cases}$$

vanishes as $\varepsilon \to 0$. In the previous section, we obtain estimate (2.34) on the Green's function of operator L^ε. We recall that Q_1 is at least quadratic in $\tilde{w}_z$ and $\tilde{R}^\varepsilon$ verifies inequalities (2.6), (2.7), and (2.8).

As in [32], [34] and [63], we use standard arguments for parabolic equations. First, we remark that local existence of a smooth solution $\tilde{w}$ for (2.38)-(2.39) is classical. Then we define

$$T^\varepsilon = \sup\{T_1 \in [0; T^*], \exists \tilde{w} \text{ solution on } \mathbb{R} \times [0; T_1), E(T_1) \leqslant 1\},$$

where

$$E(T_1) = \int_0^{T_1}\int_0^L \left(\frac{|\tilde w|}{\varepsilon^{3\gamma-\alpha}} + \frac{|\tilde w_z|}{\varepsilon^{3\gamma-\alpha-1/2}} + \frac{|\tilde w_t|}{\varepsilon^{3\gamma-2\alpha-1/2}} + \frac{|\tilde w_{zz}|}{\varepsilon^{3\gamma-3\alpha-3/2}} + \frac{|\tilde w_{tz}|}{\varepsilon^{3\gamma-2\alpha-1}} \right.$$
$$\left. + \frac{|\tilde w_{tt}|}{\varepsilon^{3\gamma-3\alpha-1}} + \frac{|\tilde w_{tzz}|}{\varepsilon^{3\gamma-4\alpha-2}} + \frac{|\tilde w_{ttz}|}{\varepsilon^{3\gamma-3\alpha-3/2}} \right) \mathrm{d}z\,\mathrm{d}t$$

with $\alpha > 0$ and $\gamma \in (2/3, 1)$, chosen later.

Before estimating the L^1-norms, we define the notation:

$$\|\tilde w\|_1 = \|\tilde w\|_{L^1((0;T^\varepsilon)\times[0,L))}, \|\tilde w\|_\infty = \|\tilde w\|_{L^\infty((0;T^\varepsilon)\times[0,L))}.$$

The continuous imbedding $W^{1,1} \subset L^\infty$ holds in z space, then

$$\|\tilde w_z\|_\infty \leqslant = \| \int_0^t \tilde w_{zt}(s) ds\|_\infty \leqslant C(\|\tilde w_z\|_1 + \|\tilde w_{ztt}\|_1 + \|\tilde w_{zzt}\|_1) \leqslant C\varepsilon^{3\gamma-4\alpha-2}$$

when $\varepsilon \leqslant 1$. Hence, $\|\tilde w\|_\infty$ tends to 0 as ε goes to 0 if γ and α are such that $\gamma - \alpha > 1/2$.

We deduce that to prove that the time existence is T^* and that we have the convergence, it remains to prove that $T^\varepsilon = T^*$ for ε small enough. In the sequel, we suppose that $T^\varepsilon < T^*$ so $E(T^\varepsilon) = 1$.

Before computing estimates on $\tilde w$ and its derivatives, let us give a useful equality on the Green function. Since L^ε is an operator with periodic coefficients, we have the L-periodicity of the Green function G^ε : for all t, τ, z, y,

$$G^\varepsilon(t, \tau, z + L, y + L) = G^\varepsilon(t, \tau, z, y).$$

Using this equality and estimate (2.34), we compute for L-periodic function ψ

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_0^{T^*}\int_0^L G^\varepsilon(t, \tau, z, y)\,\mathrm{d}z\,\mathrm{d}t \right) \psi(y)\,\mathrm{d}y \\
&\qquad = \sum_{k\in\mathbb{Z}} \int_0^L \left(\int_0^{T^*}\int_0^L G^\varepsilon(t, \tau, z, y + kL)\,\mathrm{d}z\,\mathrm{d}t \right) \psi(y + kL)\,\mathrm{d}y \\
&\qquad = \sum_{k\in\mathbb{Z}} \int_0^L \left(\int_0^{T^*}\int_0^L G^\varepsilon(t, \tau, z + kL, y + kL)\,\mathrm{d}z\,\mathrm{d}t \right) \psi(y)\,\mathrm{d}y \\
&\qquad = \int_0^L \left(\int_0^{T^*}\int_{\mathbb{R}} G^\varepsilon(t, \tau, z, y)\,\mathrm{d}z\,\mathrm{d}t \right) \psi(y)\,\mathrm{d}y.
\end{aligned}$$

Then, we can use the estimate on $\|\tilde{w}\|_\infty$ to bound $\tilde{w}$ and its derivatives. First, using (2.38)-(2.39), we have

$$\tilde{w}(t,z) = -\int_0^t \int_{\mathbb{R}} G^\varepsilon(t,\tau,z,y) \left(\tilde{R}^\varepsilon + Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z)\right)(\tau,y)\, \mathrm{d}y\, \mathrm{d}\tau.$$

Therefore, using (2.34), we deduce

$$\|\tilde{w}\|_1 \leqslant C\varepsilon^{3\gamma} + C\|\tilde{w}_z\|_\infty \|\tilde{w}_z\|_1$$

so

$$\frac{\|\tilde{w}\|_1}{\varepsilon^{3\gamma-\alpha}} \leqslant C(\varepsilon^\alpha + \varepsilon^{3\gamma-4\alpha-5/2}).$$

This can be made smaller than 1 as $\varepsilon \to 0$ if α and γ are such that $3\gamma - 4\alpha - 5/2 > 0$.

We now take the z derivative of $\tilde{w}$ and obtain an expression of $\tilde{w}_z$:

$$\tilde{w}_z(t,z) = -\int_0^t \int_{\mathbb{R}} G^\varepsilon_z(t,\tau,z,y) \left(\tilde{R}^\varepsilon + Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z)\right)(\tau,y)\, \mathrm{d}y\, \mathrm{d}\tau$$

and the estimate:

$$\frac{\|\tilde{w}_z\|_1}{\varepsilon^{3\gamma-\alpha-1/2}} \leqslant C(\varepsilon^\alpha + \varepsilon^{3\gamma-4\alpha-5/2}).$$

Differentiating equation (2.38) with respect to t, we obtain equation verified by $\tilde{w}_t$:

$$L^\varepsilon \tilde{w}_t = -\left(\tilde{R}^\varepsilon + Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z)\right)_t - \left(\frac{1}{\varphi_z}\left(df(\tilde{u}^\varepsilon_{app}) - \varphi_t + \varepsilon\frac{\varphi_{zz}}{\varphi_z^2}\right)\right)_t \cdot \tilde{w}_z + \varepsilon\left(\frac{1}{\varphi_z^2}\right)_t \tilde{w}_{zz}. \tag{2.40}$$

Using equation (2.38), we obtain

$$L^\varepsilon \tilde{w}_t - \frac{2\varphi_{zt}}{\varphi_z}\tilde{w}_t = -\left(\tilde{R}^\varepsilon + Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z)\right)_t - \frac{2\varphi_{zt}}{\varphi_z}(\tilde{R}^\varepsilon + Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z)) + l_1(\tilde{w}_z, \tilde{w}),$$

where l_1 is a continuous linear form, uniformly bounded with respect to ε. Thus, using again the Green's function and Remark 2.2, we get the inequality:

$$\|\tilde{w}_t\|_1 \leqslant C(\varepsilon^{3\gamma-1/2} + \|\tilde{w}_z\|_\infty(\|\tilde{w}_z\|_1 + \|\tilde{w}_{zt}\|_1) + \varepsilon^{3\gamma} + \|\tilde{w}_z\|_\infty\|\tilde{w}_z\|_1 + \|\tilde{w}_z\|_1 + \|\tilde{w}\|_1),$$

and we deduce

$$\frac{\|\tilde{w}_t\|_1}{\varepsilon^{3\gamma-2\alpha-1/2}} \leqslant C(\varepsilon^{2\alpha} + \varepsilon^{3\gamma-4\alpha-2}(\varepsilon^\alpha + \varepsilon^{-1/2}) + \varepsilon^{2\alpha+1/2} + \varepsilon^{3\gamma-3\alpha-2} + \varepsilon^\alpha + \varepsilon^{\alpha+1/2}).$$

Using estimates on G^ε_z, we obtain bound on $\tilde{w}_{tz}$

$$\frac{\|\tilde{w}_{tz}\|_1}{\varepsilon^{3\gamma-2\alpha-1}} \leqslant C(\varepsilon^{2\alpha} + \varepsilon^{3\gamma-4\alpha-2}(\varepsilon^\alpha + \varepsilon^{-1/2}) + \varepsilon^{2\alpha+1/2} + \varepsilon^{3\gamma-3\alpha-2} + \varepsilon^\alpha + \varepsilon^{\alpha+1/2}).$$

Using equation (2.38), we obtain bound on $\tilde{w}_{zz}$

$$\|\tilde{w}_{zz}\|_1 \leqslant \frac{C}{\varepsilon}(\|\tilde{w}_t\|_1 + \|\tilde{w}_z\|_1 + \|\tilde{w}\|_1 + \|\tilde{R}^\varepsilon\|_1 + \|Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w})\|_1),$$

$$\frac{\|\tilde{w}_{zz}\|_1}{\varepsilon^{3\gamma-3\alpha-3/2}} \leqslant C(\varepsilon^\alpha + \varepsilon^{2\alpha} + \varepsilon^{2\alpha+1/2} + \varepsilon^{3\alpha+1/2} + \varepsilon^{3\gamma-2\alpha-1}).$$

Differentiating again equation (2.40), we obtain

$$L^\varepsilon \tilde{w}_{tt} - 4\frac{\varphi_{zt}}{\varphi_z}\tilde{w}_{tt} = -\left(\tilde{R}^\varepsilon + Q_1(\tilde{u}^\varepsilon_{app}, \tilde{w}_z)\right)_{tt} + l_2(\tilde{R}^\varepsilon, Q_1, \tilde{R}^\varepsilon_t, Q_{1t}, \tilde{w}, \tilde{w}_t, \tilde{w}_z, \tilde{w}_{zt}).$$

Then, we use again (2.34) to obtain,

$$\frac{\|\tilde{w}_{tt}\|_1}{\varepsilon^{3\gamma-3\alpha-1}} \leqslant C(\varepsilon^{3\alpha} + \varepsilon^{3\gamma-4\alpha-2}(\varepsilon^{2\alpha+1/2} + \varepsilon^\alpha + \varepsilon^{-1/2}) + \varepsilon^{2\alpha+1} + \varepsilon^{\alpha+1/2} + \varepsilon^{2\alpha+1/2} + \varepsilon^\alpha),$$

and

$$\frac{\|\tilde{w}_{ttz}\|_1}{\varepsilon^{3\gamma-3\alpha-3/2}} \leqslant C(\varepsilon^{3\alpha} + \varepsilon^{3\gamma-4\alpha-2}(\varepsilon^{2\alpha+1/2} + \varepsilon^\alpha + \varepsilon^{-1/2}) + \varepsilon^{2\alpha+1} + \varepsilon^{\alpha+1/2} + \varepsilon^{2\alpha+1/2} + \varepsilon^\alpha).$$

Using again (2.40), we obtain estimate on $\tilde{w}_{zzt}$:

$$\frac{\|\tilde{w}_{tzz}\|_1}{\varepsilon^{3\gamma-4\alpha-2}} \leqslant C(\varepsilon^{4\alpha+1/2} + \varepsilon^{2\alpha+1/2} + \varepsilon^\alpha + \varepsilon^{2\alpha} + \varepsilon^{3\alpha+1/2} + \varepsilon^{\alpha+1/2}).$$

We now choose $\gamma \in (\frac{2}{3}; 1)$ and $\alpha > 0$, such that $3\gamma - 4\alpha - 5/2 > 0$. Hence, we have proved that for ε small enough, we have $E(T^\varepsilon) \leqslant \varepsilon^\beta$ with $\beta > 0$. Consequently, we can not have $T^\varepsilon < T^*$ for such an ε. Moreover, using the change of variable φ, we return to $w = u^\varepsilon - u^\varepsilon_{app}$. Thus, we have

$$\|w\|_\infty \leqslant C\varepsilon^{3\gamma-3\alpha-3/2},$$

and

$$\|u^\varepsilon - u^\varepsilon_{app}\|_\infty \leqslant C\varepsilon^{3\gamma-3\alpha-3/2}.$$

Inequality $E(T^*) \leqslant 1$ also gives

$$\|w\|_{L^\infty(L^1)} \leqslant \|w_t\|_1 \leqslant C\|\tilde{w}_t\|_1 \leqslant C\varepsilon^{3\gamma-\alpha_1/2}$$

so

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^\infty(L^1)} \leqslant C\varepsilon^{3\gamma-2\alpha-1/2}.$$

This concludes the proof of Theorem 2.3.

It only remains to prove Theorem 2.1. Since $\|u^\varepsilon_{app} - u\|_{L^\infty(L^1)} \to 0$, we have the convergence in $L^\infty((0;T^*), L^1(0;L))$. Moreover, the fast convergences of the viscous shock profiles V^j give the last point of the theorem.

2.5 Conclusion and perspectives

In this article, we have proved the persistence of solutions of the inviscid equation (2.2) close to roll-waves by adding full viscosity, with linear source term. One of the main assumptions that we have taken is the periodicity of the solution of (2.2). This one is not necessary in the construction of the approximate solution u^{ε}_{app}, but it gives that u_1 and u_2 stay bounded (because periodic). An idea to weaken this assumption would be that solution u of (2.2) approximates the roll-wave as $|x|$ goes to infinity (in particular, the shock curves would be closer as $|x|$ goes to infinity). Moreover, the periodicity of u allows us to use the method of [34] to construct the Green's function of L^{ε}. Indeed, in this step, the number of Green's functions that we consider is proportional to the number of shocks. In the periodic case, taking into account the repetitions, it returns to a finite number of periodic Green's kernels. Thus, we can write a matrix of errors, and deduce Theorem 2.4 on the existence of the Green's function relative to L^{ε} and estimates on this Green's function. Another way to hope to obtain a finite number of Green's functions could be to assume that u coincides with the roll-wave outside a bounded domain.

Furthermore, Theorem 2.1 proved here is valid in the case of an artificial viscosity. Therefore, one should also study the persistence in the case of real viscosity as presented to the system of Saint Venant (2.3).

Finally, one can also be interested in what happens in the multidimensional case. For this, we could build on work already done in the case of a single multidimensional shock, based on the study of Evans' functions at each shock [35].

Deuxième partie

Schéma de type volume fini pour les équations de Patlak-Keller-Segel

Un des grands défis actuels en mathématiques est la modélisation des phénomènes biologiques. Dans cette partie, on s'intéresse aux mouvements de cellules par interaction avec un signal chimique : le chimiotactisme. Des équations modélisant ce phénomène ont d'abord étaient posées par C. F. Patlak, E. F. Keller et L. A. Segel, puis ont mené à des systèmes plus complexes, prenant en compte davantage de paramètres. Dans un premier temps, on construit un schéma de type volume fini pour approcher les solutions du système de Patlak-Keller-Segel et on montre que ce schéma converge. Ensuite, des simulations numériques relatives à ce modèle sont données, mettant en évidence des regroupements de cellules. On adapte également le schéma à des modèles un peu plus complexes.

Chapitre 3

A new finite-volume scheme for parabolic-parabolic Patlak-Keller-Segel equations

3.1 Introduction

Chemotaxis is a process of interaction between cells, through chemical signals, secreted by the cells themselves. This type of communication between cells appears in both inflammatory and immunologic processes, as in growth processes (colonies of bacteria, development of an embryo,...). In the case of Patlak-Keller-Segel model, the density of cells n evolves by different processes: diffusion, and interaction with chemoattractant. The chemical concentration c also moves by diffusion process, and it is produced by the cells and degraded over time. So Patlak-Keller-Segel system reads

$$\begin{cases} \partial_t n = \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega, t \in \mathbb{R}^+, & (3.1) \\ \varepsilon \partial_t c = \Delta c + n - \alpha c, & x \in \Omega, t \in \mathbb{R}^+, & (3.2) \\ \nabla n \cdot \nu = 0, \quad \nabla c \cdot \nu = 0, & x \in \partial\Omega, t \in \mathbb{R}^+, & (3.3) \\ n(t=0) = n_0, & x \in \Omega, & (3.4) \\ c(t=0) = c_0 \text{ if } \varepsilon > 0, & x \in \Omega, & (3.5) \end{cases}$$

under its adimensionalized form and with Neuman boundary conditions. Here, α, ε are non-negative constants and Ω a domain of $\mathbb{R}^d$. We refer to [41, 59] for the derivation of this model.

The interest of this system is to model a phenomenon of aggregation of cells when the total mass of cells is over a critical mass. This phenomena also depends on the space dimension. In particular, in dimension $d = 2$, and when $\varepsilon = \alpha = 0$, the phenomena has been precisely analyzed: there is existence of a global solution for $M := \int_\Omega n_0(x)\,\mathrm{d}x$ under a critical mass and blow-up in finite time for M over the same critical mass [8, 10]. See also the survey [39] and the references therein.

The case where $\varepsilon > 0$ is less studied because involving other difficulties. Indeed, since the model with $\varepsilon = 0$ is parabolic-elliptic (the first equation (3.1) is parabolic and the second one (3.2) becomes elliptic), it can be resumed to a single parabolic equation, with nonlocal nonlinear flux. Here, both equations are parabolic and strongly coupled. In the case of bounded domain, local existence for all total mass is already proved [8]. There is also results of global existence for subcritical mass (with critical mass $M_0 = 4\pi$ for smooth boundary) [50, 29]. In the case of whole space ($\Omega = \mathbb{R}^d$), there is also an existence result for subcritical mass [15]. Using energy estimates, it is proved that for non-negative (n_0, c_0) such that $M < 8\pi$, there exists global weak solution (n, c) to (3.1)–(3.5). Concerning surcritical mass, it is proved that self-similar solutions exists for all total mass [9]. But there is no general result for supercritical mass.

This article focused on the discretization by finite-volume method of Patlak-Keller-Segel system for $\varepsilon \neq 0$ and the convergence of the obtained scheme for dimension $d = 2$. For the parabolic-elliptic system, finite-volume methods [27, 64], a finite element method [48], and a finite difference method [70] have already been proposed. In the case of parabolic-parabolic Patlak-Keller-Segel system, discontinuous Galerkin method [25] and finite volume scheme [16] are already developed. Since the second equation in Patlak-Keller-Segel system is not conservative, A. Kurganov et al. chose to first differentiate the second equation (3.2). Then, the finite volume method is designed with splitting x-derivatives and y-derivatives, $(x, y) \in \Omega$, that means that they rewrite the system as

$$\partial_t U + \partial_x F(U) + \partial_y G(U) = \Delta U + R(U)$$

and treat separately the space variables and derivatives: each flux (F and G) are treated depending on the hyperbolicity of the system defined with this flux, i.e. the global hyperbolicity of the system is not taken into account.

Actually, the differentiated system is not hyperbolic, while $n > 0$. In our scheme, we also use the differentiated system but we separate the two equations. The first one is treated as a transport equation with upwind flux and the other one is discretized using centered method. Moreover, we prove in this paper that our scheme converges to a weak solution of the differentiated system and we give some numerical computation for Patlak-Keller-Segel system and two other models: a chemotaxis model with source terms and a haptotaxis model.

The paper is organized as follows. In the following section, we introduced the differentiated system and we develop some properties of it, in particular energy estimate. We also define here a notion of weak solution. In Section 3.3, after defining the notation and the scheme, we give the main result of the article: the convergence of the scheme towards a weak solution of (3.6)–(3.10) under a small mass assumption. Section 3.4 is devoted to the convergence of the scheme. A discrete energy estimate gives bounds on the unknowns, uniform with respect to the spatial scale of the discretization. We deduce the convergence of the scheme to a weak solution. In the last section, we give the results of our numerical computation.

3.2 Derivation of the model: continuous analysis

The purpose of this paper is to compute numerical simulation on parabolic-parabolic Patlak-Keller-Segel model (3.1)–(3.5). As in [16] and [25], we first differentiate the second equation (3.2) with respect to space variable. So, we note $S = (r\ s)^T := \nabla c$ and we rewrite system (3.1)–(3.5) as:

$$\begin{cases} \partial_t n = \Delta n - \nabla \cdot (nS), & (x,y) \in \Omega, t \in \mathbb{R}^+, & (3.6)\\ \varepsilon \partial_t S = \Delta S + \nabla n - \alpha S, & (x,y) \in \Omega, t \in \mathbb{R}^+, & (3.7)\\ \nabla \times S = 0, & (x,y) \in \Omega, t \in \mathbb{R}^+, & (3.8)\\ \nabla n \cdot \nu = 0, \quad S \cdot \nu = 0, & (x,y) \in \partial\Omega, t \in \mathbb{R}^+, & (3.9)\\ (n,S)(t=0) = (n_0, S_0), & (x,y) \in \Omega. & (3.10) \end{cases}$$

Before giving the numerical scheme, we give some properties of this continuous system.

First, we remark that we can write (3.6)–(3.10) as a system of advection-reaction-diffusion equations:

$$\partial_t U + \nabla \cdot F(U) = D\Delta U + R(U)$$

where

$$U = \begin{pmatrix} n \\ r \\ s \end{pmatrix},\ F(U) = \begin{pmatrix} nr & ns \\ -n/\varepsilon & 0 \\ 0 & -n/\varepsilon \end{pmatrix},\ D = \operatorname{diag}(1, \varepsilon^{-1}, \varepsilon^{-1}),\ R(U) = -\alpha \begin{pmatrix} 0 \\ r \\ s \end{pmatrix}.$$

Under this form, it is natural to wonder about the hyperbolicity of the "purely" convective system. Since we search for positive n, the eigenvalues of

$$\omega_1 \begin{pmatrix} r & n & 0 \\ -1/\varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \omega_2 \begin{pmatrix} s & 0 & n \\ 0 & 0 & 0 \\ -1/\varepsilon & 0 & 0 \end{pmatrix}$$

for $\omega_1, \omega_2 \in \mathbb{R}$ such that $\omega_1^2 + \omega_2^2 = 1$ are not real for all ω_1, ω_2. Therefore, the "purely convective part" of system (3.6)-(3.7) is not hyperbolic while $n > 0$.

However, thanks to diffusion terms, we get stability and the following energy estimates:

Proposition 3.1. *For all smooth (n, S) classical solution of (3.6)–(3.10) such that n is non-negative, we have the estimates:*

– *mass conservation*

$$\int_\Omega n(t, x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega n_0(x, y)\,\mathrm{d}x\,\mathrm{d}y = M, \tag{3.11}$$

– *and energy estimate*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(n \log n + \varepsilon \frac{|S|^2}{2}\right) \\ &\qquad \leqslant -\left(4 - \frac{2C_\Omega M}{\delta}\right)\int_\Omega |\nabla\sqrt{n}|^2 - (1-\delta)\int_\Omega |\nabla S|^2 - \alpha\int_\Omega |S|^2 + \frac{C_\Omega}{\delta} M^2 \end{aligned} \tag{3.12}$$

for all $0 < \delta < 1$, and where $C_\Omega > 0$ only depends on Ω.

Proof. Let (n, S) be a smooth solution of (3.6)–(3.10) on $(0; T), T > 0$. First, integrating (3.6) on Ω, we obtain mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n(t, x, y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Therefore, n verifies equality (3.11).

Multiplying equation (3.6) by $1 + \log n$, and integrating on Ω, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n\log n &= -\int_\Omega \frac{1}{n}|\nabla n|^2 + \int_\Omega \nabla n\cdot S,\\ &= -4\int_\Omega |\nabla\sqrt{n}|^2 - \int_\Omega n\nabla\cdot S.\end{aligned} \tag{3.13}$$

Now, multiplying equation (3.7) by S and integrating, we get

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|S|^2}{2} = -\int_\Omega |\nabla S|^2 - \int_\Omega n\nabla\cdot S - \alpha\int_\Omega |S|^2. \tag{3.14}$$

Summing (3.13) and (3.14), and applying Hölder and Young inequalities to $\int_\Omega n\nabla\cdot S$, we obtain for all $\delta > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(n\log n + \varepsilon\frac{|S|^2}{2}\right) \leqslant -4\int_\Omega |\nabla\sqrt{n}|^2 - \int_\Omega |\nabla S|^2 + \frac{2}{\delta}\int_\Omega n^2 + \frac{\delta}{2}\int_\Omega |\nabla\cdot S|^2 - \alpha\int_\Omega |S|^2.$$

We now use Gagliardo-Nirenberg-Sobolev inequality [37]

$$\int_\Omega n^2 \leqslant C_\Omega\int_\Omega n\int_\Omega |\nabla\sqrt{n}|^2 + C_\Omega\left(\int_\Omega n\right)^2 = C_\Omega\left(M\int_\Omega |\nabla\sqrt{n}|^2 + M^2\right) \tag{3.15}$$

and estimate on the divergence

$$\int_\Omega |\nabla\cdot S|^2 \leqslant 2\int_\Omega |\nabla S|^2$$

to obtain (3.12). □

We can compare this energy estimate with that relative to parabolic-parabolic Patlak-Keller-Segel system (3.1)-(3.2) [15, Proposition 2.1]. Here, we obtain estimates on S in $L^\infty((0;T), L^2(\Omega))$ and ∇S in $L^2(\Omega_T)$ while V. Calvez and L. Corrias only estimate gradient $S = \nabla c$ in $L^\infty((0;T), L^2(\Omega))$. So, we need more regularity.

The purpose of this article is to give a numerical scheme and convergence to a weak solution of (3.6)–(3.10). In the sequel, we suppose that the domain is the torus $\Omega = \mathbb{T}$ and the boundary conditions are periodic, which does not change the previous estimates. Noting $\Omega_T = (0;T)\times\Omega$, we define a notion of weak solution of (3.6)–(3.10):

Definition 3.1. *$(n, S) \in L^2(\Omega_T)\times L^2((0;T); H^1(\Omega))$ is a* weak solution *of (3.6)–(3.10) on $(0;T)$ if for all $\varphi \in \mathcal{C}_c^3([0;T)\times\Omega), \psi \in \mathcal{C}_c^2([0;T)\times\Omega)$,*

$$\int_{\Omega_T} n\partial_t\varphi + \int_{\Omega_T} n\Delta\varphi + \int_{\Omega_T} nS\cdot\nabla\varphi + \int_\Omega n_0\varphi(t=0) = 0, \tag{3.16}$$

$$\varepsilon\int_{\Omega_T} S\cdot\partial_t\psi - \int_{\Omega_T}\nabla S\cdot\nabla\psi - \int_{\Omega_T} n\nabla\cdot\psi - \alpha\int_{\Omega_T} S\cdot\psi + \int_\Omega S_0\cdot\psi(t=0) = 0. \tag{3.17}$$

3.3 Numerical scheme and main result

In this section, we derive a scheme for the new formulation (3.6)–(3.10) of the Patlak-Keller-Segel equations. We use here a finite volume approach for the discretization in space variable. The convective term in the first equation (3.6) is discretized as a conservative transport term: n with the speed S, so we use central-upwind method. In the second equation (3.7), we use centered method to discretize the convective term.

For the sake of simplicity, we only consider here spatial discretization, with constant scale. First, we introduce the notations: $h = (\Delta x, \Delta y)$ the small spatial scale, $x_\beta = \beta\Delta x, y_\gamma = \gamma\Delta y$ and the cells $C_{i,j} = [x_{i-1/2}; x_{i+1/2}] \times [y_{j-1/2}; y_{j+1/2}]$. We also note that $h > 0$ will mean that $\Delta x, \Delta y > 0$. We define the approximations $n^0_{i,j}$ and $S^0_{i,j}$ of the initial datum n_0, S_0 at the point (x_i, y_j) as usual by

$$n^0_{i,j}\Delta x\Delta y = \int_{C_{i,j}} n_0(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad S^0_{i,j}\Delta x\Delta y = \int_{C_{i,j}} S_0(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.18}$$

Denoting by $n_{i,j}$ and $S_{i,j}$ the approximations of the mean values of n and S at the point (x_i, y_j), the numerical scheme studied in this paper reads

$$\frac{\mathrm{d}n_{i,j}}{\mathrm{d}t} = \frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y}, \tag{3.19}$$

$$\varepsilon\frac{\mathrm{d}S_{i,j}}{\mathrm{d}t} = \frac{H_{i+1/2,j} - H_{i-1/2,j}}{\Delta x} + \frac{J_{i,j+1/2} - J_{i,j-1/2}}{\Delta y} - \alpha S_{i,j} \tag{3.20}$$

where $F_{i+1/2,j}, G_{i,j+1/2}$ are the upwind flux for the first equation

$$F_{i+1/2,j} = \frac{n_{i+1,j} - n_{i,j}}{\Delta x} - (r^+_{i+1/2,j} n_{i,j} - r^-_{i+1/2,j} n_{i+1,j}),$$
$$G_{i,j+1/2} = \frac{n_{i,j+1} - n_{i,j}}{\Delta y} - (s^+_{i,j+1/2} n_{i,j} - s^-_{i,j+1/2} n_{i,j+1}),$$

with

$$a^+ = \max(a, 0), \qquad a^- = -\min(a, 0),$$

and $H_{i+1/2,j}, J_{i,j+1/2}$ are the centered flux for the second equation:

$$H_{i+1/2,j} = \left(\frac{r_{i+1,j} - r_{i,j}}{\Delta x} + n_{i+1/2,j}, \quad \frac{s_{i+1,j} - s_{i,j}}{\Delta x}\right)^T,$$
$$J_{i,j+1/2} = \left(\frac{r_{i,j+1} - r_{i,j}}{\Delta y}, \quad \frac{s_{i,j+1} - s_{i,j}}{\Delta y} + n_{i,j+1/2}\right)^T.$$

We have also used the notations

$$\begin{aligned}
r_{i+1/2,j} &= \frac{r_{i+1,j}+r_{i,j}}{2},\\
s_{i,j+1/2} &= \frac{s_{i,j+1}+s_{i,j}}{2},\\
n_{i+1/2,j} &= \frac{n_{i+1,j}+n_{i,j}}{2},\\
n_{i,j+1/2} &= \frac{n_{i,j+1}+n_{i,j}}{2}.
\end{aligned}$$

Now, we define some functions, which are numerical approximations of (n, S):

$$n_h(t,x,y) = n_{i,j}(t) \quad \text{if } (x,y) \in C_{i,j}, \tag{3.21}$$

$$S_h(t,x,y) = S_{i,j}(t) \quad \text{if } (x,y) \in C_{i,j}. \tag{3.22}$$

We also define the approximations of gradient and Laplacian of a function u by

$$\nabla_h u(x,y) = \left(\frac{u(x,y)-u(x-\Delta x)}{\Delta x}, \quad \frac{u(x,y)-u(x,y-\Delta y)}{\Delta y}\right)^T, \tag{3.23}$$

$$\begin{aligned}
\Delta_h u(x,y) =& \frac{u(x+\Delta x,y)+u(x-\Delta x,y)-2u(x,y)}{\Delta x^2}\\
&+\frac{u(x,y+\Delta y)+u(x,y-\Delta y)-2u(x,y)}{\Delta y^2}.
\end{aligned} \tag{3.24}$$

Using all these notations, we will prove in the sequel that (n_h, S_h) defined by the scheme (3.19)-(3.20) exists and converges as $h = (\Delta x, \Delta y)$ goes to zero to a weak solution of new formulation (3.6)-(3.7) of Patlak-Keller-Segel equations in the sense of Definition 3.1.

Theorem 3.1. *Let $(n_0, S_0) \in L^2(\Omega)$ such that*

$$\begin{aligned}
&n_0(x,y) > 0 \qquad \forall (x,y) \in \Omega,\\
&\int_\Omega n_0(x,y)\,\mathrm{d}x\,\mathrm{d}y < \frac{2}{K_\Omega}.
\end{aligned}$$

Let (n_h, S_h) be the solution of the scheme (3.19)-(3.20) with the discrete initial data n_h^0, S_h^0 defined by (3.18).

Then, there exist nonnegative function n and function $S = (r\ s)^T$ such that for all $T > 0$, n belongs to $L^2(\Omega_T)$ and S belongs to $L^2((0;T), H^1(\Omega))$ and there is a *subsequence of* (n_h, S_h) *which satisfies the convergences*

$$\begin{aligned} n_h &\rightharpoonup n && \text{weakly in } L^2(\Omega_T), \\ n_h &\to n && \text{strongly in } L^2((0;T), L^1(\Omega)), \\ S_h &\to S && \text{strongly in } L^2(\Omega_T), \\ \nabla_h S_h &\rightharpoonup \nabla S && \text{weakly in } L^2(\Omega_T) \end{aligned}$$

as the spatial scale h *goes to zero.*

Moreover, for all $T < \infty$, (n, S) *is a weak solution of (3.6)–(3.10) on* $(0;T)$.

To prove this theorem, we first give a discrete energy estimate, similar to (3.12). We deduce some uniform bounds on n_h and S_h, which will give convergence of subsequences to a limit (n, S). Then, rewriting (3.19)-(3.20) in a weak form and passing to the limit we obtain that the limits n and S are weak solution of differentiated Patlak-Keller-Segel equation (3.6)–(3.10).

3.4 Convergence

The aim of this section is to prove that the solutions of the scheme (3.19)-(3.20) converge as the spatial scale h goes to 0. An energy estimate gives us uniform bounds, which suffice to prove strong compactness.

3.4.1 Discrete energy estimate

As in the continuous case, we have mass conservation and an energy estimate:

Proposition 3.2. *Let* $h > 0$, (n_h^0, S_h^0) *constant on each* $C_{i,j}$, *with values* $\in \mathbb{R}^3$. *Then, there exist* $T > 0$ *and a unique* $(n_h, S_h) \in \mathcal{C}^1([0;T])$ *classical solution of (3.19)-(3.20) on* $[0;T]$. *Moreover, this solution verifies the conservation of positivity:*

$$\text{if } n_h^0 > 0, \text{ then, } \forall t, n_h(t) > 0, \tag{3.25}$$

the mass conservation:

$$\int_\Omega n_h(t, x, y)\, \mathrm{d}x\, \mathrm{d}y = \sum_{i,j} \Delta x \Delta y\ n_{i,j}^0 =: M_h \tag{3.26}$$

and the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(n_h \log n_h + \varepsilon \frac{S_h^2}{2}\right) \leqslant -\left(4 - \frac{2K_\Omega M_h}{\delta}\right)\int_\Omega |\nabla_h \sqrt{n_h}|^2 \\ - (1-\delta)\int_\Omega |\nabla_h S_h|^2 - \alpha \int_\Omega |S_h|^2 + \frac{2}{\delta}\frac{M_h^2}{|\Omega|} \quad (3.27)$$

for all $\delta > 0$, and where C only depends on Ω and $|\Omega|$ is the Lebesgue measure of Ω.

Moreover, if $M_h < \frac{2}{K_\Omega}$, we have global existence: $T = +\infty$.

Proof. First, we remark that Cauchy-Lipschitz theorem gives the well-posedness of equations (3.19)-(3.20) on $[0;T]$ where $T > 0$. Moreover, integrating (3.19), we get mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i,j} \Delta x \Delta y \; n_{i,j}(t)\right) = 0.$$

Furthermore, we can rewrite (3.19) under the form

$$\frac{\mathrm{d}}{\mathrm{d}t} n_{i,j} = -\alpha_{i,j} n_{i,j} + b_{i,j}(n_{i-1,j}, n_{i+1,j}, n_{i,j-1}, n_{i,j+1}),$$

where

$$\alpha_{i,j} = \frac{1}{\Delta x}\left(r^+_{i+1/2,j} + r^-_{i-1/2,j} + \frac{2}{\Delta x}\right) + \frac{1}{\Delta y}\left(s^+_{i,j+1/2} + s^-_{i,j-1/2} + \frac{2}{\Delta y}\right),$$

and

$$b_{i,j}(\dots) = \frac{n_{i+1,j}}{\Delta x}\left(r^-_{i+1/2,j} + \frac{1}{\Delta x}\right) + \frac{n_{i-1,j}}{\Delta x}\left(r^+_{i-1/2,j} + \frac{1}{\Delta x}\right) \\ + \frac{n_{i,j+1}}{\Delta y}\left(s^-_{i,j+1/2} + \frac{1}{\Delta y}\right) + \frac{n_{i,j-1}}{\Delta y}\left(s^+_{i,j-1/2} + \frac{1}{\Delta y}\right).$$

Since $b_{i,j}(...)$ remains positive as long as $n_{i-1,j}, n_{i+1,j}, n_{i,j-1}, n_{i,j+1}$ are, we deduce that $n_{i,j}$ remains positive.

To prove the energy estimate (3.27), we adapt the proof done in the continuous case. First, we multiply (3.19) by $1+\log n_h$ and we integrate in space. Using periodic boundary condition, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_h \log n_h \,\mathrm{d}x\,\mathrm{d}y\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i,j} \Delta x \Delta y \; n_{i,j} \log n_{i,j}\right) \\ = -\sum_{i,j} \Delta x \Delta y \left(\frac{\log n_{i,j} - \log n_{i-1,j}}{\Delta x} F_{i-1/2,j} + \frac{\log n_{i,j} - \log n_{i,j-1}}{\Delta y} G_{i,j-1/2}\right) \\ = E_1 + E_2 + E_3 + E_4,$$

where

$$
\begin{aligned}
E_1 &= -\sum_{i,j} \Delta x \Delta y \; \frac{\log n_{i,j} - \log n_{i-1,j}}{\Delta x} \frac{n_{i,j} - n_{i-1,j}}{\Delta x}, \\
E_2 &= \sum_{i,j} \Delta x \Delta y \; \frac{\log n_{i,j} - \log n_{i-1,j}}{\Delta x} (r^+_{i-1/2,j} n_{i-1,j} - r^-_{i-1/2,j} n_{i,j}), \\
E_3 &= -\sum_{i,j} \Delta x \Delta y \; \frac{\log n_{i,j} - \log n_{i,j-1}}{\Delta y} \frac{n_{i,j} - n_{i,j-1}}{\Delta y}, \\
E_4 &= \sum_{i,j} \Delta x \Delta y \; \frac{\log n_{i,j} - \log n_{i,j-1}}{\Delta y} (s^+_{i,j-1/2} n_{i,j-1} - s^-_{i,j-1/2} n_{i,j}).
\end{aligned}
$$

Using the inequality

$$
(\log x - \log y)(x - y) \geqslant 4(\sqrt{x} - \sqrt{y})^2,
$$

we have

$$
E_1 \leqslant -4 \sum_{i,j} \Delta x \Delta y \left(\frac{\sqrt{n_{i,j}} - \sqrt{n_{i-1,j}}}{\Delta x} \right)^2
$$

Computing a similar inequality for E_3, we deduce

$$
E_1 + E_3 \leqslant -4 \int_\Omega |\nabla_h \sqrt{n_h}|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{3.28}
$$

We now take $\tilde{n}_{i-1/2,j} \in (n_{i,j}, n_{i-1,j})$ such that

$$
\frac{\log n_{i,j} - \log n_{i-1,j}}{\Delta x} = \frac{1}{\Delta x} \frac{n_{i,j} - n_{i-1,j}}{\tilde{n}_{i-1/2,j}}.
$$

Since $\tilde{n}_{i-1/2,j} \in (n_{i,j}, n_{i-1,j})$, we obtain

$$
\frac{n_{i,j} - n_{i-1,j}}{\Delta x} \left(r^+_{i-1/2,j} \frac{n_{i-1,j}}{\tilde{n}_{i-1/2,j}} - r^-_{i-1/2,j} \frac{n_{i,j}}{\tilde{n}_{i-1/2,j}} \right) \leqslant \frac{n_{i,j} - n_{i-1,j}}{\Delta x} (r^+_{i-1/2,j} - r^-_{i-1/2,j}),
$$

so that

$$
E_2 \leqslant \sum_{i,j} \Delta x \Delta y \; \frac{n_{i,j} - n_{i-1,j}}{\Delta x} r_{i-1/2,j} = -\sum_{i,j} \Delta x \Delta y \; n_{i,j} \frac{r_{i+1/2,j} - r_{i-1/2,j}}{\Delta x},
$$

and similarly

$$
E_4 \leqslant -\sum_{i,j} \Delta x \Delta y \; n_{i,j} \frac{s_{i,j+1/2} - s_{i,j-1/2}}{\Delta y}.
$$

Furthermore, multiplying (3.20) by $S_{i,j}$, we obtain

$$
\begin{aligned}
\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{|S_h|^2}{2}\,\mathrm{d}x\,\mathrm{d}y\right) &= \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i,j}\Delta x\Delta y\ \frac{|S_{i,j}|^2}{2}\right)\\
&= -\sum_{i,j}\Delta x\Delta y\ \left(\frac{S_{i,j}-S_{i-1,j}}{\Delta x}H_{i-1/2,j}+\frac{S_{i,j}-S_{i,j-1}}{\Delta y}J_{i,j-1/2}+\alpha|S_{i,j}|^2\right)\\
&= D_1+D_2-\alpha\int_\Omega |S_h|^2\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

with

$$
\begin{aligned}
D_1 &= -\sum_{i,j}\Delta x\Delta y\ \left(\left|\frac{S_{i,j}-S_{i-1,j}}{\Delta x}\right|^2+\left|\frac{S_{i,j}-S_{i,j-1}}{\Delta y}\right|^2\right) = -\int_\Omega |\nabla_h S_h|^2\,\mathrm{d}x\,\mathrm{d}y,\\
D_2 &= -\sum_{i,j}\Delta x\Delta y\ \left(\frac{r_{i,j}-r_{i-1,j}}{\Delta x}n_{i-1/2,j}+\frac{s_{i,j}-s_{i,j-1}}{\Delta y}n_{i,j-1/2}\right),\\
&= -\sum_{i,j}\Delta x\Delta y\ \left(\frac{r_{i+1/2,j}-r_{i-1/2,j}}{\Delta x}+\frac{s_{i,j+1/2}-s_{i,j-1/2}}{\Delta y}\right)n_{i,j}.
\end{aligned}
$$

Since $E_2+E_4\leqslant D_2$, to obtain (3.27) it only remains to bound D_2. Now, using Hölder and Young inequalities, we obtain for all $\delta>0$

$$
\begin{aligned}
D_2 &\leqslant \frac{1}{\delta}\sum_{i,j}\Delta x\Delta y\ n_{i,j}^2+\frac{\delta}{4}\sum_{i,j}\Delta x\Delta y\ \left|\frac{r_{i+1/2,j}-r_{i-1/2,j}}{\Delta x}+\frac{s_{i,j+1/2}-s_{i,j-1/2}}{\Delta y}\right|^2\\
&\leqslant \frac{1}{\delta}\int_\Omega n_h^2\,\mathrm{d}x\,\mathrm{d}y+\frac{\delta}{2}\int_\Omega |\nabla_h S_h|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Furthermore, we can apply discrete Gagliardo-Nirenberg-Sobolev inequality [27, Lemma 3.2]

$$
\int_\Omega n_h^2 \leqslant \frac{M_h^2}{|\Omega|}+K_\Omega M_h\int_\Omega \left|\nabla_h\sqrt{n_h}\right|^2. \tag{3.29}
$$

Now, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(n_h\log n_h+\varepsilon\frac{|S_h|^2}{2}\right) &= -\,\alpha\int_\Omega |S_h|^2+(E_1+E_3)+D_1+(E_2+E_4+D_2)\\
&\leqslant -\,\alpha\int_\Omega |S_h|^2-4\int_\Omega|\nabla_h\sqrt{n_h}|^2-\int_\Omega|\nabla_h S_h|^2\\
&\quad+\frac{2}{\delta}\left(\frac{M_h^2}{|\Omega|}+K_\Omega M_h\int_\Omega|\nabla_h\sqrt{n_h}|^2\right)+\delta\int_\Omega|\nabla_h S_h|^2.
\end{aligned}
$$

$\square$

We remark that the discrete energy is similar to the continuous energy (3.12). Using notations (3.21)–(3.24) and discrete energy estimates (3.26)-(3.27), we obtain bounds on $n_h, S_h, \nabla_h\sqrt{n_h}, \nabla_h S_h$.

Proposition 3.3. *Let $(n_0, S_0) \in L^2(\Omega)$ such that n_0 is non negative and $M := \int_\Omega n_0 < \frac{2}{K_\Omega}$, and $(n_h^0, S_h^0)_{h>0}$ defined by (3.18). Then, for $h > 0$ small enough, there exists a unique $(n_h, S_h) \in \mathcal{C}^1(\mathbb{R}^+ \times \Omega)$ classical solution of (3.19)-(3.20) on $\mathbb{R}^+$. For all $T > 0$, this solution verifies the following bounds, uniform in h:*

$$n_h \text{ is bounded in } L^\infty((0;T), L^1(\Omega)) \text{ (mass conservation)}, \tag{3.30}$$

$$n_h \log n_h \text{ is bounded in } L^\infty((0;T), L^1(\Omega)), \tag{3.31}$$

$$\nabla_h\sqrt{n_h} \text{ is bounded in } L^2(\Omega_T), \tag{3.32}$$

$$S_h \text{ is bounded in } L^\infty((0;T), L^2(\Omega)), \tag{3.33}$$

$$\nabla_h S_h \text{ is bounded in } L^2(\Omega_T). \tag{3.34}$$

Proof. Let $T > 0$. Since $n_0 > 0$, and $M_h < \frac{2}{K_\Omega}$ for small enough space scale h, we can chose $\delta > 0$ such that $1 - \delta > 0$ and $4 - \frac{2K_\Omega M_h}{\delta} > 0$. Then, estimate (3.27) and mass conservation (3.26) give the uniform bounds. □

3.4.2 Convergence

In this section, we prove convergence for the solution of the ordinary equations (3.19)-(3.20) with initial data (3.18). Using the bounds given in Proposition 3.3, we easily get weak convergence. Since system (3.6)-(3.7) is nonlinear, we also need strong convergence. To obtain it, we need bounds on the time derivatives [66]. First, we give that bounds. Next, we prove the convergence of the scheme.

Lemma 3.4. *Let $(n_0, S_0) \in L^2(\Omega)$ such that n_0 is non negative and $M := \int_\Omega n_0 < \frac{2}{K_\Omega}$, $(n_h^0, S_h^0)_{h>0}$ defined by (3.18), and the classical solution $(n_h, S_h)_{h>0} \in \mathcal{C}^1(\mathbb{R}^+ \times \Omega)$ of (3.19)-(3.20) on $\mathbb{R}^+$. Then, for all $T > 0$,*

$$\frac{\mathrm{d}n_h}{\mathrm{d}t} \text{ is uniformly bounded in } L^2((0;T), H^{\sigma+1}(\Omega)'), \tag{3.35}$$

$$\frac{\mathrm{d}S_h}{\mathrm{d}t} \text{ is uniformly bounded in } L^2((0;T), H^{\sigma}(\Omega)'), \tag{3.36}$$

with $\sigma > 2$.

Proof. Let $\varphi \in H^{\sigma+1}(\Omega)$. Using the definition of n_h and integrating in space,

$$\int_{\Omega_T} \frac{\mathrm{d}n_h}{\mathrm{d}t}\varphi = \int_0^T \sum_{i,j} \frac{\mathrm{d}n_{i,j}}{\mathrm{d}t} \int_{C_{i,j}} \varphi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \sum_{i,j} \Delta x\Delta y\ \frac{\mathrm{d}n_{i,j}}{\mathrm{d}t}(t)\varphi_{i,j}(t)\,\mathrm{d}t,$$

where we note $\varphi_{i,j}(t) = \dfrac{1}{|C_{i,j}|}\displaystyle\int_{C_{i,j}} \varphi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x$. Then, thanks to equation (3.19), we obtain

$$\begin{aligned}\int_{\Omega_T} \frac{\mathrm{d}n_h}{\mathrm{d}t}\varphi = &\int_0^T \sum_{i,j} \Delta x \Delta y \left(\frac{\varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{i,j}}{\Delta x^2} + \frac{\varphi_{i,j+1} + \varphi_{i,j-1} - 2\varphi_{i,j}}{\Delta y^2} \right) n_{i,j}\,\mathrm{d}t \\ &+ \int_0^T \sum_{i,j} \Delta x \Delta y\ \frac{\varphi_{i,j} - \varphi_{i-1,j}}{\Delta x} (r^+_{i-1/2,j} n_{i-1,j} - r^-_{i-1/2,j} n_{i,j})\,\mathrm{d}t \\ &+ \int_0^T \sum_{i,j} \Delta x \Delta y\ \frac{\varphi_{i,j} - \varphi_{i,j-1}}{\Delta y} (s^+_{i,j-1/2} n_{i,j-1} - s^-_{i,j-1/2} n_{i,j})\,\mathrm{d}t.\end{aligned}$$

Since n_h is bounded in $L^1(\Omega_T)$, and n_h and S_h are uniformly bounded in $L^2(\Omega_T)$, we obtain the bound

$$\int_{\Omega_T} \frac{\mathrm{d}n_h}{\mathrm{d}t}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leqslant M_h T \sup_{\Omega_T} |\Delta_h \varphi_h| + 4\|n_h\|_{L^2}\|S_h\|_{L^2} \sup_{\Omega_T} |\nabla_h \varphi_h|.$$

Then, it suffices to choose σ such that discrete derivatives of φ are bounded by $\|\varphi\|_{H^{\sigma+1}}$. Now, we have

$$\begin{aligned}|\nabla_h \varphi_h| &= \max\left(\left| \int_{C_{i,j}} \frac{\varphi(\cdot) - \varphi(\cdot - \Delta x, \cdot)}{\Delta x} \frac{\mathrm{d}x\,\mathrm{d}y}{|C_{i,j}|} \right|, \left| \int_{C_{i,j}} \frac{\varphi(\cdot) - \varphi(\cdot, \cdot - \Delta y)}{\Delta y} \frac{\mathrm{d}x\,\mathrm{d}y}{|C_{i,j}|} \right| \right) \\ &\leqslant \|\nabla\varphi\|_{L^\infty}\end{aligned}$$

and similarly

$$|\Delta_h \varphi_h| \leqslant 2\|\nabla^2 \varphi\|_{L^\infty}.$$

Thus, if $\sigma + 1 > 2 + 2/2$, there exists C which only depends on the uniform bounds on n_h, S_h such that for all $\varphi \in H^{\sigma+1}(\Omega)$

$$\left| \int_\Omega \frac{\mathrm{d}n_h}{\mathrm{d}t}\varphi\,\mathrm{d}x\,\mathrm{d}y \right| \leqslant C\|\varphi\|_{H^{\sigma+1}}.$$

This property implies that $\frac{\mathrm{d}n_h}{\mathrm{d}t}$ is uniformly bounded in $L^2((0;T), H^{\sigma+1}(\Omega)')$.

Similarly, we obtain the uniform bound for $\frac{\mathrm{d}S_h}{\mathrm{d}t}$ in $L^2((0;T), H^{\sigma}(\Omega)')$. $\square$

Using all this bounds, we prove the convergence part of Theorem 3.1.

Proof of Theorem 3.1: convergences. As seen in Proposition 3.3, n_h is uniformly bounded in $L^\infty((0;T), L^1(\Omega))$. Moreover, the bound on $\nabla_h\sqrt{n_h}$ in $L^2(\Omega_T)$ gives uniform bound on $\nabla_h n_h$ in $L^1(\Omega_T)$. Indeed, since $n_h \geqslant 0$,

$$\left| \frac{n_{i,j} - n_{i-1,j}}{\Delta x} \right| = (\sqrt{n_{i,j}} + \sqrt{n_{i-1,j}}) \left| \frac{\sqrt{n_{i,j}} - \sqrt{n_{i-1,j}}}{\Delta x} \right|.$$

Thus, we have the uniform bound

$$
\begin{aligned}
\|\nabla_h n_h\|_{L^1} = \int_\Omega |\nabla_h n_h| \,\mathrm{d}x\,\mathrm{d}y &= \sum_{i,j} \Delta x \Delta y \ \left(\left| \frac{n_{i,j} - n_{i-1,j}}{\Delta x} \right| + \left| \frac{n_{i,j} - n_{i,j-1}}{\Delta y} \right| \right) \\
&\leqslant \int_\Omega n_h \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega |\nabla_h \sqrt{n_h}|^2 \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

To use the partial compactness result of J. Simon [66], we define the functional spaces

$$
\begin{aligned}
X_n &= \left\{ n \in L^1(\Omega) : \sup_\eta \left\| \frac{n(\cdot+\eta) - n(\cdot)}{|\eta|} \right\|_{L^1(\Omega)} < +\infty \right\}, \\
B_n &= L^1(\Omega), \qquad Y_n = H^{\sigma+1}(\Omega)', \sigma > 2.
\end{aligned}
$$

Then, we have the bounds

$$
\begin{aligned}
&n_h \text{ is bounded in } L^1((0;T), X_n), \\
&n_h \text{ is bounded in } L^\infty((0;T), B_n), \\
&\frac{\mathrm{d}n_h}{\mathrm{d}t} \text{ is bounded in } L^1((0;T), Y_n)
\end{aligned}
$$

Since the embedding $X_n \subset B_n$ is compact, we can use the compactness result [66] showing that there exist a subsequence of n_h and $n \in L^2((0;T), B_n)$ such that

$$
n_h \to n \quad \text{strongly in } L^2((0;T), L^1(\Omega)).
$$

Moreover, using again discrete Gagliardo-Nirenberg-Sobolev (3.29), n_h is uniformly bounded in $L^2(\Omega_T)$. Thus, $n \in L^2(\Omega_T)$ and there exists a subsequence of n_h (still labeled n_h) which verifies the weak convergence:

$$
n_h \rightharpoonup n \quad \text{weakly in } L^2(\Omega_T).
$$

We now recall the bounds on S_h

$$
\begin{aligned}
&S_h \text{ is bounded in } L^\infty((0;T), L^2(\Omega)), \\
&\nabla_h S_h \text{ is bounded in } L^2(\Omega_T), \\
&\frac{\mathrm{d}S_h}{\mathrm{d}t} \text{ is bounded in } L^2((0;T), H^\sigma(\Omega)').
\end{aligned}
$$

As for n_h, we use again compactness result [66] to conclude on the existence of a subsequence of S_h (still labeled S_h) which strongly converges in $L^2(\Omega_T)$. Indeed, using the notations

$$X = \left\{ S \in L^2(\Omega) : \sup_\eta \left\| \frac{S(\cdot + \eta) - S(\cdot)}{|\eta|^{1/2}} \right\|_{L^2(\Omega)} < +\infty \right\},$$
$$B = L^2(\Omega), \qquad Y = H^\sigma(\Omega)',$$

we have that

$$S_h \text{ is bounded in } L^2((0;T), X),$$
$$\frac{\mathrm{d}S_h}{\mathrm{d}t} \text{ is bounded in } L^2((0;T), Y).$$

We remark that we have the compact embedding $X \subset B$ and the inclusion $B \subset Y$. So, using again Simon result, there exist $S \in L^2((0;T), B)$ and a subsequence of S_h such that

$$S_h \to S \quad \text{strongly in } L^2(\Omega_T)$$

Finally, since $\nabla_h S_h$ is bounded in $L^2(\Omega_T)$, there exists $\Theta \in L^2(\Omega_T)$ such that the following weak convergence holds for a subsequence

$$\nabla_h S_h \rightharpoonup \Theta \text{ weakly in } L^2(\Omega_T).$$

Furthermore, $\Theta = \nabla S$ is true in disrtributional sense. Then, we have proved all the convergences of Theorem 3.1. It only remains to prove that (n, S) verifies equations (3.6)–(3.10) in the weak sense (3.16)-(3.17). □

3.4.3 Weak solution

In the previous subsection, we proved that, up to the extraction of a subsequence, the approximate solution (n_h, S_h) converges to (n, S). We now prove that (n, S) is a weak solution of differentiated Patlak-Keller-Segel equations (3.6)–(3.10).

Proposition 3.5. *The functions n, S defined as limits of n_h, S_h in Theorem 3.1 satisfy equation (3.6)–(3.10) in the weak sense (3.16)-(3.17).*

Proof. First, we state that (n_h, S_h) verifies a weak formulation of (3.18)–(3.20). Indeed, multiplying the equations by φ, ψ and using discrete integration by parts, we have that for all $\varphi \in \mathcal{C}^3([0;T) \times \Omega), \psi = (\psi^1, \psi^2) \in \mathcal{C}^2([0;T) \times \Omega)$,

$$\int_{\Omega_T} n_h \left(\partial_t \varphi + \Delta_h \varphi\right) + \int_{\Omega_T} (nS)_h \nabla_h \varphi + \int_\Omega n_h^0 \varphi(t=0) = 0, \tag{3.37}$$

where

$$(nr)_h(\cdot) = \left(\frac{r_h(\cdot) + r_h(\cdot - \Delta x, \cdot)}{2}\right)^+ n_h(\cdot - \Delta x, \cdot) - \left(\frac{r_h(\cdot) + r_h(\cdot - \Delta x, \cdot)}{2}\right)^- n_h(\cdot),$$

$$(ns)_h(\cdot) = \left(\frac{s_h(\cdot) + s_h(\cdot, \cdot - \Delta y)}{2}\right)^+ n_h(\cdot, \cdot - \Delta y) - \left(\frac{s_h(\cdot) + s_h(\cdot, \cdot - \Delta y)}{2}\right)^- n_h(\cdot),$$

and

$$\int_{\Omega_T} (\varepsilon S_h \cdot \partial_t \psi - \nabla_h S_h \cdot \nabla_h \psi - \alpha S_h \cdot \psi) - \int_{\Omega_T} n_h \frac{\psi^1(\cdot + \Delta x, \cdot) - \psi^1(\cdot - \Delta x, \cdot)}{2\Delta x} - \int_{\Omega_T} n_h \frac{\psi^2(\cdot, \cdot + \Delta y) - \psi^2(\cdot, \cdot - \Delta y)}{2\Delta y} + \int_{\Omega} \varepsilon S_h^0 \cdot \psi(t=0) = 0. \quad (3.38)$$

Now, it only remains to prove that n, S are weak solution of (3.6)-(3.7). Let $\psi = (\psi^1, \psi^2) \in \mathcal{C}_c^2([0;T) \times \Omega)$. Since $S_h \to S$ strongly in $L^2(\Omega_T)$, we have the convergence

$$\int_{\Omega_T} (\varepsilon S_h \cdot \partial_t \psi - \alpha S_h \cdot \psi) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \to \int_{\Omega_T} (\varepsilon S \cdot \partial_t \psi - \alpha S \cdot \psi) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t. \quad (3.39)$$

Moreover, we have

$$\int_{\Omega} \varepsilon S_h^0 \cdot \psi(t=0) \, \mathrm{d}x \, \mathrm{d}y \to \int_{\Omega} \varepsilon S_0 \cdot \psi(t=0) \, \mathrm{d}x \, \mathrm{d}y. \quad (3.40)$$

To prove (3.17), it only remains to treat term

$$\int_{\Omega_T} \nabla_h S_h \cdot \nabla_h \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t,$$

the other ones would be treated in the same way. We can write

$$\int_{\Omega_T} \nabla_h S_h \cdot \nabla_h \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t = \int_{\Omega_T} \nabla_h S_h \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t + \int_{\Omega_T} \nabla_h S_h \cdot (\nabla_h \psi - \nabla \psi) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t.$$

Using the weak convergence

$$\nabla_h S_h \rightharpoonup \nabla S \text{ in } L^2(\Omega_T),$$

we obtain

$$\int_{\Omega_T} \nabla_h S_h \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \to \int_{\Omega_T} \nabla S \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t. \quad (3.41)$$

Using Hölder inequality, we bound the other integral

$$\int_{\Omega_T} \nabla_h S_h \cdot (\nabla_h \psi - \nabla \psi) \leqslant \left(\int_{\Omega_T} |\nabla_h S_h|^2 \right)^{1/2} \left(\int_{\Omega_T} |\nabla_h \psi - \nabla \psi|^2 \right)^{1/2},$$

where $\nabla_h S_h$ and $\nabla_h \psi - \nabla \psi$ are uniformly bounded in $L^2(\Omega_T)$. Indeed,

$$\int_{\Omega_T} \left(\frac{\psi(\cdot) - \psi(\cdot - \Delta x, \cdot)}{\Delta x} - \partial_x \psi(\cdot) \right)^2 \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \leqslant C \Delta x^2 T \|\nabla^2 \psi\|_{L^\infty}^2 \to 0$$

as Δx goes to 0. So, combining (3.38) with the convergences (3.39),(3.40) and (3.41), we get weak formulation (3.17).

We proceed by the same way to obtain that n verifies weak formulation (3.16). The main difference states in the nonlinearity of equation (3.6). Let $\varphi \in \mathcal{C}_c^3([0;T) \times \Omega)$. We split (3.37) in

$$\begin{aligned}
H_1 &= \int_{\Omega_T} n_h \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t + \int_\Omega n_h \varphi(t=0) \, \mathrm{d}x \, \mathrm{d}y, \\
H_2 &= \int_{\Omega_T} n_h \Delta_h \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t, \\
H_{21} &= \int_{\Omega_T} n_h \frac{\varphi(\cdot + \Delta x, \cdot) + \varphi(\cdot - \Delta x, \cdot) - 2\varphi(\cdot)}{\Delta x^2} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t, \\
H_{22} &= \int_{\Omega_T} n_h \frac{\varphi(\cdot, \cdot + \Delta y) + \varphi(\cdot, \cdot - \Delta y) - 2\varphi(\cdot)}{\Delta y^2} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t, \\
H_3 &= \int_{\Omega_T} (nS)_h \cdot \nabla_h \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t.
\end{aligned}$$

Then, using weak convergence of n_h in $L^2(\Omega_T)$, we get

$$H_1 \to \int_{\Omega_T} n \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t + \int_\Omega n_0 \varphi(t=0) \, \mathrm{d}x \, \mathrm{d}y. \tag{3.42}$$

Now, we can write

$$H_{21} = \int_{\Omega_T} n_h \partial_x^2 \varphi + \int_{\Omega_T} n_h \left(\frac{\varphi(\cdot + \Delta x, \cdot) + \varphi(\cdot - \Delta x, \cdot) - 2\varphi(\cdot)}{\Delta x^2} - \partial_x^2 \varphi(\cdot) \right).$$

Then, using again the weak convergence of n_h in $L^2(\Omega_T)$, the convergence

$$H_{21} \to \int_{\Omega_T} n \partial_x^2 \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \tag{3.43}$$

is equivalent to the convergence to zero of the second integral. Now, since $\varphi \in \mathcal{C}^3$ and using Hölder inequality, the following bound holds

$$\int_{\Omega_T} n_h \left(\frac{\varphi(\cdot + \Delta x, \cdot) + \varphi(\cdot - \Delta x, \cdot) - 2\varphi(\cdot)}{\Delta x^2} - \partial_x^2 \varphi(\cdot) \right) \leqslant C \Delta x \|n_h\|_{L^2} \|\nabla^3 \varphi\|_{L^\infty} \to 0$$

as Δx goes to zero. We use the same method to prove the convergence

$$H_{22} \to \int_{\Omega_T} n \partial_y^2 \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t. \tag{3.44}$$

Now, it only remains to prove that

$$H_3 \to \int_{\Omega_T} nS \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t.$$

First, we have

$$\left(\frac{r_h(\cdot) + r_h(\cdot - \Delta x, \cdot)}{2} \right)^- \to r^- \qquad \text{strongly in } L^2(\Omega_T),$$

and

$$n_h \rightharpoonup n \qquad \text{weakly in } L^2(\Omega_T),$$

then we have the weak convergence

$$\left(\frac{r_h(\cdot) + r_h(\cdot - \Delta x, \cdot)}{2} \right)^- n_h(\cdot) \rightharpoonup r^- n \qquad \text{weakly in } L^1(\Omega_T).$$

Now,

$$\int_{\Omega_T} (nr)_h \frac{\varphi(\cdot) - \varphi(\cdot - \Delta x, \cdot)}{\Delta x} = \int_{\Omega_T} (nr)_h \partial_x \varphi + \int_{\Omega_T} (nr)_h \left(\frac{\varphi(\cdot) - \varphi(\cdot - \Delta x, \cdot)}{\Delta x} - \partial_x \varphi(\cdot) \right)$$

and

$$\left| \frac{\varphi(\cdot) - \varphi(\cdot - \Delta x, \cdot)}{\Delta x} - \partial_x \varphi(\cdot) \right| \leqslant \frac{\Delta x}{2} \|\nabla^2 \varphi\|_{L^\infty} \to 0$$

as Δx goes to zero. Therefore, we obtain the convergence

$$\begin{aligned} \int_{\Omega_T} (nr)_h \frac{\varphi(\cdot) - \varphi(\cdot - \Delta x, \cdot)}{\Delta x} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t &\to \int_{\Omega_T} (r^+ n - r^- n) \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \\ &= \int_{\Omega_T} rn \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t. \end{aligned}$$

Then,

$$H_3 = \int_{\Omega_T} (nS)_h \nabla_h \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \to \int_{\Omega_T} nS \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t. \tag{3.45}$$

Finally, (3.37), (3.42)–(3.45) give that (n, S) verifies (3.16). □

These proposition concludes the proof of Theorem 3.1.

3.5 Numerical simulations

In this section, we give some numerical results and we compare our results to that of A. Chertock and A. Kurganov in [16]. We apply our scheme to both parabolic-parabolic Patlak-Keller-Segel system and two related problems: one more complicated chemotaxis model and the haptotaxis model. For parabolic-parabolic Patlak-Keller-Segel system, we first consider the case of subcritical mass, and then the case of supercritical mass.

In all numerical examples, we use periodic boundary conditions in space variables and second order Runge-Kutta solver for the time evolution. To keep stability and positivity in our numerical simulations, we adapt the time step at each step. Indeed, we ask it to verify the CFL conditions

$$\Delta t \leqslant \min\left(\frac{\Delta x}{8\max(|r_{i,j}|)}, \frac{\Delta y}{8\max(|s_{i,j}|)}, \frac{\Delta x^2}{8}, \frac{\Delta y^2}{8}\right).$$

Moreover, to improve the approximation, we use here a slope-limiter method, with van Leer limiter function.

3.5.1 Patlak-Keller-Segel model

In this section, we perform numerical simulations for Patlak-Keller-Segel system written in the form (3.6)–(3.10) in dimension $d = 2$ and for $\varepsilon = 1$. Several initial data are considered to compare subcritical and supercritical cases. When the total mass of cells is less than a critical mass, there exists a global solution for (3.6)–(3.10), so we expect to observe a uniform distribution of the cells. For total mass over a critical mass, we expect to observe the formation of a singularity.

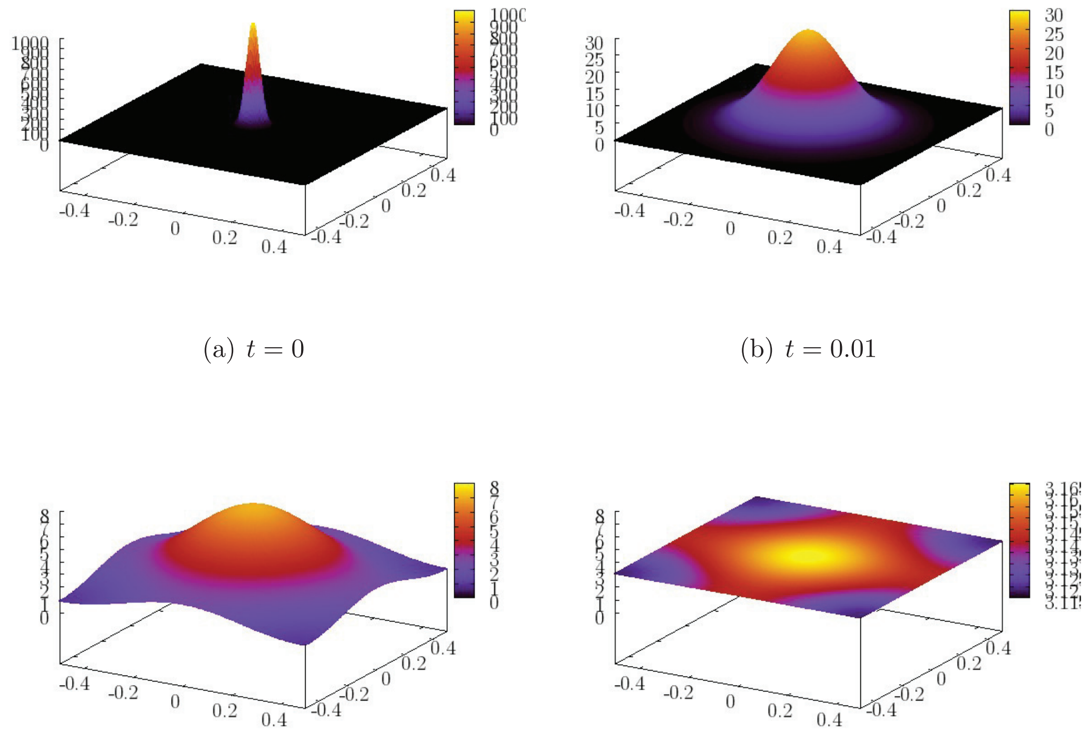
Subcritical mass

We first consider Patlak-Keller-Segel system on the square domain $[-0.5; 0.5] \times [-0.5; 0.5]$ with the subcritical initial concentration of cells

$$n_0(x, y) = 1000\ e^{-1000(x^2+y^2)},$$

and the initial concentration of chemoattractant c

$$c_0(x, y) = 0,$$



(a) $t = 0$ (b) $t = 0.01$

(c) $t = 0.04$ (d) $t = 0.2$

Figure 3.1: Solution n of (3.6) by our scheme with $\Delta x = \Delta y = 1/201$ for subcritical mass $M \sim \pi$.

so that

$$S_0(x, y) = 0.$$

Since the total mass of cells is around π, there exists a global weak solution: this seems to go to a stationary solution: the constant profile, where the constant is the total mass.

In Figure 3.1, we plot the cell concentration n on the square domain at the times $t = 0, t = 0.01, t = 0.04, t = 0.2$ using our scheme (3.19)-(3.20). We observe that cells, initially concentrated around 0, are distributed on the square with the time, to finally be uniformly distributed.

Supercritical mass

In this article, we have proved that, for subcritical mass, there exists global weak solution for (3.6)–(3.10) and that the solutions of scheme (3.18)–(3.20) converge to such a solution. The existence result is similar to the result of [15] for the parabolic-parabolic Patlak-Keller-Segel system (3.1)–(3.5), where the critical mass is 8π. But, there is not general result for supercritical mass. Numerically, we see that it seems to be a phenomenon of aggregation of cells whose presence could not depend on the initial concentration of chemoattractant: this concentration would have only influence on the time of aggregation.

Supercritical mass with nonzero c
First, we discretize Patlak-Keller-Segel equations for supercritical mass and nonzero concentration of chemoattractant. We use here

$$n_0(x,y) = 10000\ e^{-100(x^2+y^2)}, \qquad c_0(x,y) = 5000\ e^{-5(x^2+y^2)},$$

so that

$$S_0(x,y) = -50000\ e^{-5(x^2+y^2)}\ (x\ y)^T$$

on the square domain $[-0.5;0.5]\times[-0.5;0.5]$. Then, the mass of cells is around 100π which is larger than the critical mass 8π.

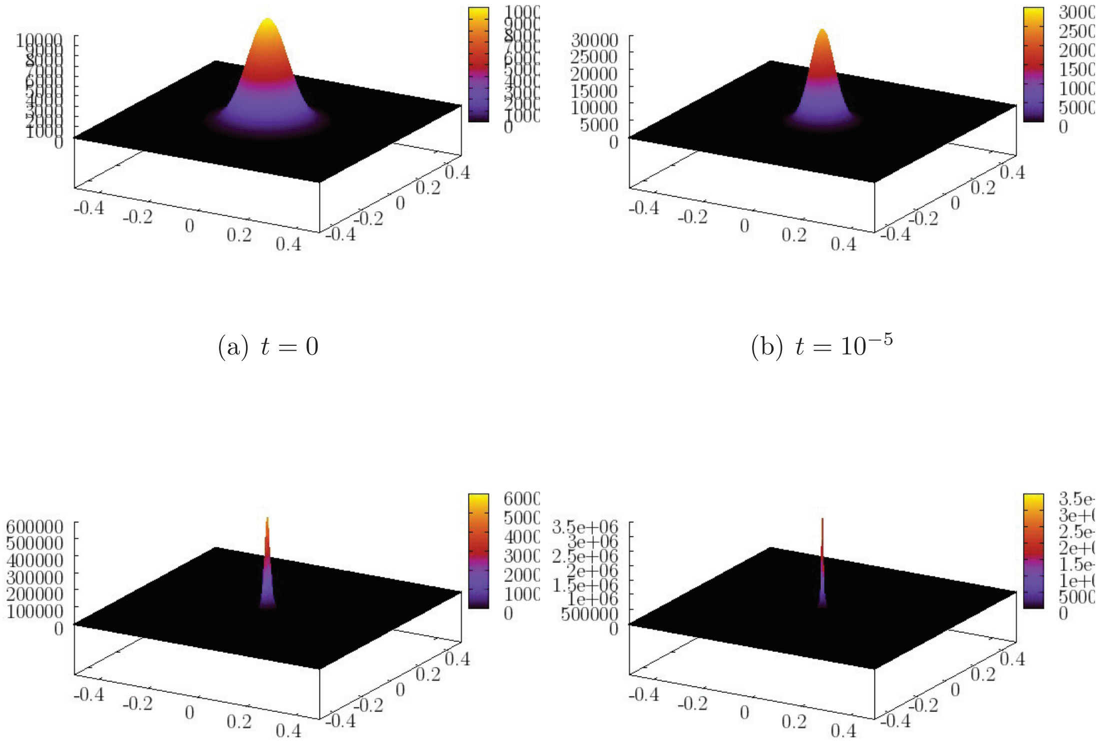
We observed by numerical simulations that n seems to become singular in 0 (Dirac function) between times $t = 4\times10^{-5}$ and $t = 5\times10^{-5}$. Indeed, for $t < 4\times10^{-5}$, the numerical solution seems to converge to a state $N(t,x,y)$ as the space scale goes to 0. Instead, this convergence does not seem to persist at $t = 5\times10^{-5}$: over the space scale is fine, the maximum concentration is high. We can see in Figure 3.2 the formation of the singularity: we plot the cell concentration n for times $t = 0, t = 10^{-5}, t = 4\times10^{-5}, t = 5\times10^{-5}$ and a space scale $\Delta x = \Delta y = 1/401$.

Supercritical mass with $c = 0$
We will see here that the aggregation phenomenon also happens in case where the chemical concentration is zero, even if it is delayed. Indeed, with the same initial cell concentration than in the previous simulation

$$n_0(x,y) = 10000\ e^{-100(x^2+y^2)},$$

and zero initial concentration of chemoattractant

$$c_0(x,y) = 0,$$



(a) $t = 0$

(b) $t = 10^{-5}$

(c) $t = 4 \times 10^{-5}$

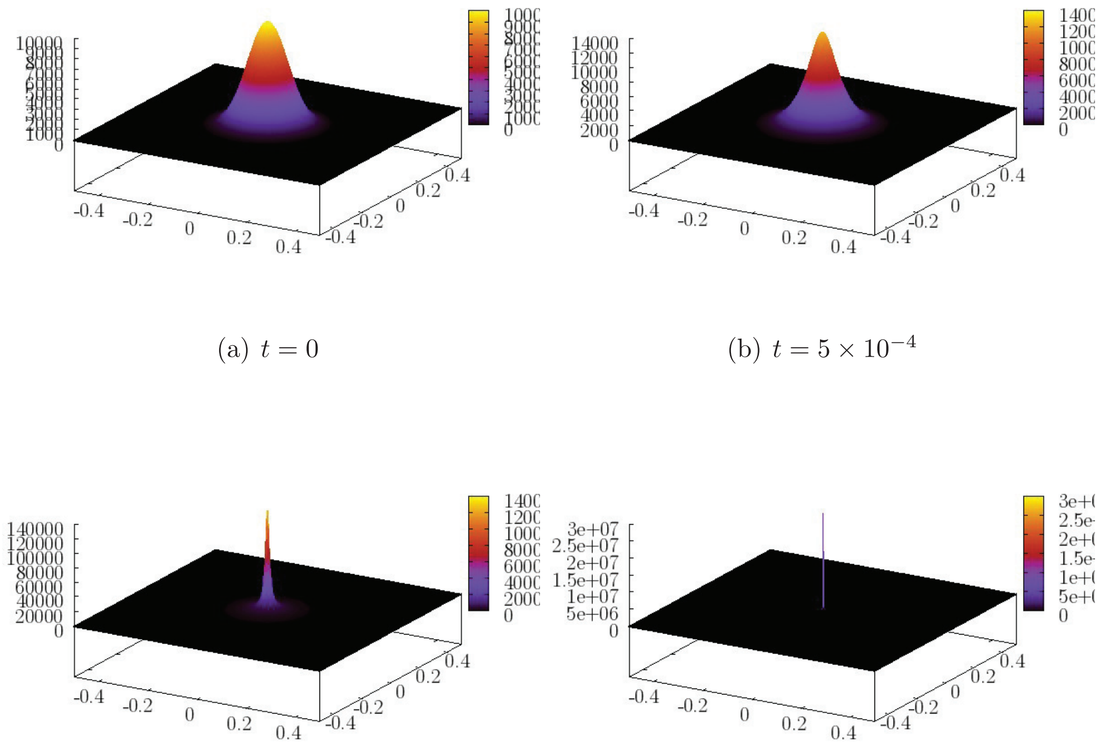
(d) $t = 5 \times 10^{-5}$

Figure 3.2: Singularity on n for supercritical mass $M \sim 100\pi$ and nonzero initial chemoattracant concentration with $\Delta x = \Delta y = 1/401$.

so that

$$S_0(x, y) = 0$$

on the square domain $[-0.5; 0.5] \times [-0.5; 0.5]$, we obtain a singularity at time between 10^{-3} and 2×10^{-3}, while in the previous simulation the singularity time is between $4 \times 10^{-5}, 5 \times 10^{-5}$. This formation of singularity is plotted in Figure 3.3: for times $t = 0, t = 5 \times 10^{-4}, t = 10^{-3}, t = 2 \times 10^{-3}$, we plot the cell concentration n on the square domain. As in the previous simulations, we observe that, for $t < 10^{-3}$, the solutions of the scheme seem to converge as the scale is refined, and for $t = 2 \times 10^{-3}$, the solutions found by the scheme seem to be unbounded.



(a) $t = 0$

(b) $t = 5 \times 10^{-4}$

(c) $t = 10^{-3}$

(d) $t = 2 \times 10^{-3}$

Figure 3.3: Singularity on n for supercritical mass $M \sim 100\pi$ and zero initial chemoattractant concentration with $\Delta x = \Delta y = 1/401$.

3.5.2 Other models

To complete numerical simulations of chemotaxis, we adapt the scheme to two other related models. The first one is a more realistic chemotaxis model, whereas the other one is modeling haptotaxis phenomenon. The particularity of the systems studied in this part is that we get more unknowns, and we keep the equation on the chemoattractant c.

As in the previous simulations, we consider square domain with periodic boundary conditions, and apply RK2 for the time discretization.

Model of chemotactic bacteria patterns in liquid medium

Here, we consider a mathematical model of chemotaxis for which the nonlinearity depends on c and ∇c and the nutrient concentration is taken into account in the production of chemoattractant. More specifically, it describes bacteria patterns in a liquid medium that contains sufficient nutrients for the bacteria, that means that the nutrient concentration is assumed to be constant [69].

So, we consider the following equations

$$\begin{cases} \partial_t n = \Delta n - a\nabla \cdot \left(\dfrac{n}{(1+c)^2} \nabla c \right), & x \in \Omega, t \in \mathbb{R}^+, \\ \varepsilon \partial_t c = d_c \Delta c + \dfrac{n^2}{1+n^2}, & x \in \Omega, t \in \mathbb{R}^+, \\ n(t=0) = n_0, \quad c(t=0) = c_0, & x \in \Omega, \end{cases}$$

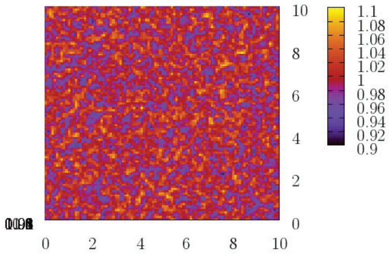
where the diffusion coefficient d_c, and the coefficient a are positive constants. As for Patlak-Keller-Segel system, we differentiate the second equation with respect to x and y and rewrite the system as follows

$$\begin{cases} \partial_t n = \Delta n - a\nabla \cdot \left(\frac{n}{(1+c)^2} S \right), & x \in \Omega, t \in \mathbb{R}^+, & (3.46) \\ \varepsilon \partial_t c = d_c \Delta c + \frac{n^2}{1+n^2}, & x \in \Omega, t \in \mathbb{R}^+, & (3.47) \\ \varepsilon \partial_t S = d_c \Delta S - \nabla \left(\frac{1}{1+n^2} \right), & x \in \Omega, t \in \mathbb{R}^+, & (3.48) \\ n(t=0) = n_0, \quad c(t=0) = c_0, \quad S(t=0) = S_0, & x \in \Omega, & (3.49) \end{cases}$$

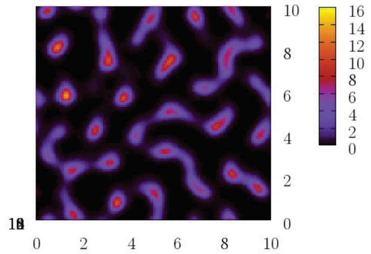
where $S := \nabla c$.

We now use our scheme to discretize this model. First, we denote by $c_{i,j}$ an approximation of the mean value of c on the cell $C_{i,j}$. Then, the numerical scheme reads

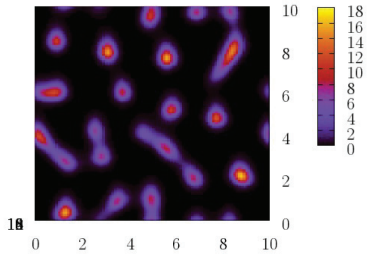
$$\begin{aligned} \frac{\mathrm{d}n_{i,j}}{\mathrm{d}t} &= \frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y}, \\ \varepsilon \frac{\mathrm{d}c_{i,j}}{\mathrm{d}t} &= d_c \frac{c_{i+1,j} + c_{i-1,j} - 2c_{i,j}}{\Delta x^2} + d_c \frac{c_{i,j+1} + c_{i,j-1} - 2c_{i,j}}{\Delta y^2} + \frac{n_{i,j}^2}{1+n_{i,j}^2}, \\ \varepsilon \frac{\mathrm{d}S_{i,j}}{\mathrm{d}t} &= \frac{H_{i+1/2,j} - H_{i-1/2,j}}{\Delta x} + \frac{J_{i,j+1/2} - J_{i,j-1/2}}{\Delta y}, \end{aligned}$$



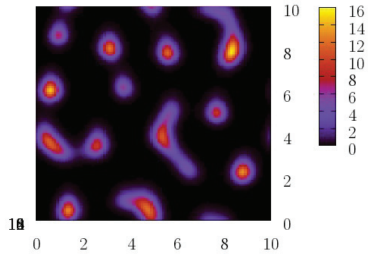
(a) $t = 0$



(b) $t = 1$



(c) $t = 2$



(d) $t = 3$

Figure 3.4: Solution n of (3.46)–(3.49) by our scheme with $\Delta x = \Delta y = 1/10$.

where

$$
\begin{aligned}
F_{i+1/2,j} &= -a(r^+_{i+1/2,j} n_{i,j} - r^-_{i+1/2,j} n_{i+1,j})/(1 + c^2_{i,j}) + \frac{n_{i+1,j} - n_{i,j}}{\Delta x}, \\
G_{i,j+1/2} &= -a(s^+_{i,j+1/2} n_{i,j} - s^-_{i,j+1/2} n_{i,j+1})/(1 + c^2_{i,j}) + \frac{n_{i,j+1} - n_{i,j}}{\Delta y}, \\
r_{i+1/2,j} &= \frac{r_{i+1,j} + r_{i,j}}{2}, \\
s_{i,j+1/2} &= \frac{s_{i,j+1} + s_{i,j}}{2}, \\
H_{i+1/2,j} &= -\frac{1}{2}\left(\frac{1}{1 + n_{i+1,j}} + \frac{1}{1 + n_{i,j}}\right) + d_c \frac{S_{i+1,j} - S_{i,j}}{\Delta x}, \\
J_{i,j+1/2} &= -\frac{1}{2}\left(\frac{1}{1 + n_{i,j+1}} + \frac{1}{1 + n_{i,j}}\right) + d_c \frac{S_{i,j+1} - S_{i,j}}{\Delta y}.
\end{aligned}
$$

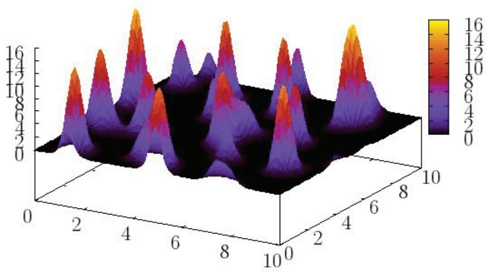


Figure 3.5: Solution n of (3.46)–(3.49) by our scheme with $\Delta x = \Delta y = 1/10$ at $t = 3$.

As in [16], we applied this scheme with the following random initial data:

$$n_0(x, y) = 0.9 + 0.2\sigma(x, y),$$

where σ is a random variable uniformly distributed on $[0; 1]$ and (x, y) is in the square domain $[0; 10] \times [0; 10]$. The initial concentration of chemoattractant is

$$c_0(x, y) = 0,$$

and the parameters are chosen as

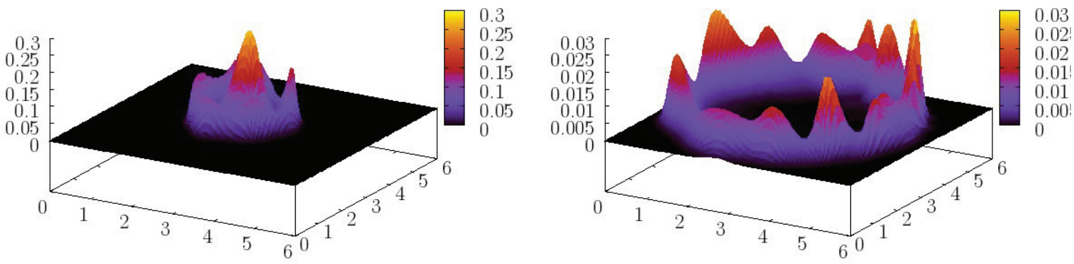
$$a = 80, d_c = 0.33, \varepsilon = 1.$$

We plot in Figures 3.4, 3.5 the results of numerical simulations. In Figure 3.4, we plot the cell concentration n on the square. At $t = 0$, the initial data is a uniform distribution of cells around 1. At $t = 1$, cells begin to cluster. But we observe that there is not formation of singularity, even after $t = 3$. These results match well with experimental data from [13, 14] and numerical results from [69] and [16].

A haptotaxis model

To conclude, we apply our scheme to a haptotaxis model

$$\begin{cases} \partial_t n = d\Delta n - \nabla \cdot (\chi(c) n \nabla c) - n + \rho(w) n, & x \in \Omega, t \in \mathbb{R}^+, & (3.50) \\ \varepsilon \partial_t c = -amc, & x \in \Omega, t \in \mathbb{R}^+, & (3.51) \\ \partial_t m = d\Delta m + n - \beta m, & x \in \Omega, t \in \mathbb{R}^+, & (3.52) \\ \partial_t w = d\Delta w + \gamma c - w - \eta(n) w, & x \in \Omega, t \in \mathbb{R}^+, & (3.53) \\ n(t=0) = n_0, \quad c(t=0) = c_0, & x \in \Omega, & (3.54) \end{cases}$$



(a) $t = 2$ (b) $t = 10$

Figure 3.6: Solution n of (3.50)–(3.54) by our scheme with $\Delta x = \Delta y = 1/201$.

where n is the tumor cells density, c the density of extracellular matrix macromolecules, m the concentration of matrix degradative enzyme, and w the concentration of oxygen. We will specify the parameters in the sequel, with the numerical computation. This model is a simplified version of one proposed in [2]. In the case of bounded domain Ω and with Neuman boundary conditions, C. Walker and G. Webb prove the existence of global solution for nonnegative initial conditions [72] and with assumptions on the parameters.

To apply our scheme, we take the gradient of the second equation. There is no new difficulty here so we do not rewrite the derivation and the scheme, we only give the choice of parameters, for which the theorem of C. Walker and G. Webb holds

$$
\begin{aligned}
&d = 0.01, \varepsilon = 1,\\
&a = 5,\ \beta = 0.01,\ \gamma = 5,\\
&\chi(c) = 0.4,\ \rho(w) = \frac{2w}{1+w},\ \eta(n) = \frac{2n}{1+n},
\end{aligned}
$$

and initial data:

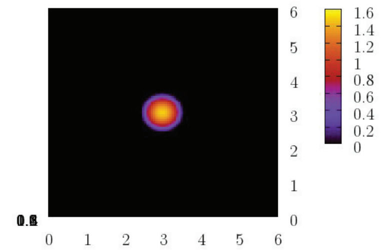
$$
\begin{aligned}
&n_0(x,y) = 5\left(0.3 - (x-3)^2 - (y-3)^2\right)^+,\\
&c_0(x,y) = 0.05\cos\left(\frac{5\pi x^2}{18}\right)\sin\left(\frac{13\pi y^2}{72}\right),\\
&m_0(x,y) = n_0(x,y), \quad w_0(x,y) = 4c_0(x,y),
\end{aligned}
$$

for (x,y) in the square domain $[0;6]\times[0;6]$. We remark that computations (Figure 3.7) match with the results of [16].

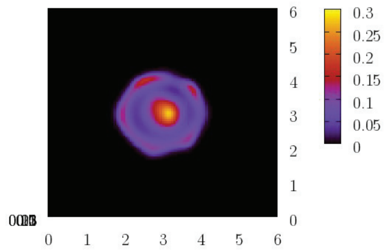
3.6 Conclusion

Here, we proposed and studied a finite-volume scheme for Patlak-Keller-Segel system written in the form (3.6)–(3.7). For simplicity reasons, we only consider periodic boundary conditions, for which we have shown that the scheme converges to a weak solution of (3.6)–(3.7) as space scale tends to 0. To improve the approximation, slope limiters have been used in numerical simulations, but on one hand the convergence has not been demonstrated with them and secondly, we currently have no estimate on the speed of convergence.

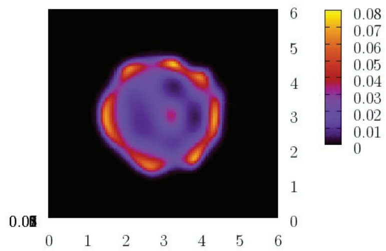
We also give some numerical computations for the Patlak-Keller-Segel system and for related models already studied by A. Chertock and A. Kurganov [16]: an other chemotaxis model and a haptotaxis model. This results seems to match with biological results. Moreover, we can see in [27, 16] that numerical computations could give some other patterns, which match with biological experiments [13, 14].



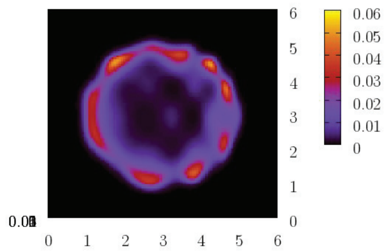
(a) $t = 0$



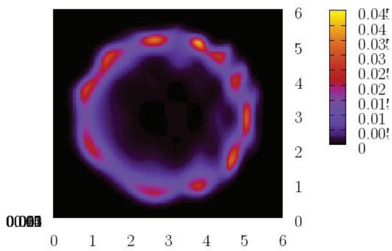
(b) $t = 2$



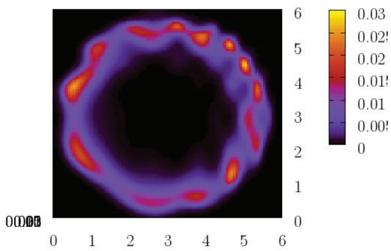
(c) $t = 4$



(d) $t = 6$



(e) $t = 8$



(f) $t = 10$

Figure 3.7: Solution n of (3.50)–(3.54) by our scheme with $\Delta x = \Delta y = 1/201$.

Annexe

A été mis en annexe un article issu d'un travail fait en 2004 en collaboration avec Pierre Degond [7]. Cette partie traite de la modélisation du trafic routier sur une route. Après avoir justifié la mise en place d'un nouveau modèle, on montre que les équations obtenues ont bien des solutions, au sens faible.

Berthelin, F., Degond, P., Le Blanc, V., Moutari, S., Rascle, M., and Royer, J. A traffic-flow model with constraints for the modeling of traffic jams. *Math. Models Methods Appl. Sci. 18*, suppl. (2008), 1269–1298.

Annexe A

A traffic-flow model with constraints for the modeling of traffic jams

Abstract. Recently, F. Berthelin et al, [6] introduced a traffic flow model describing the formation and the dynamics of traffic jams. This model which consists of a Constrained Pressureless Gas Dynamics system assumes that the maximal density constraint is independent of the velocity. However, in practice, the distribution of vehicles on a highway depends on their velocity. In this paper we propose a more realistic model namely the *Second Order Model with Constraints (in short SOMC)*, derived from the A. Aw & M. Rascle model, [3], which takes into account this feature. Moreover, when the maximal density constraint is saturated, the SOMC model "relaxes" to the M. J. Lighthill & G. B. Whitham model, [47]. We prove an existence result of weak solutions for this model by means of cluster dynamics in order to construct a sequence of approximations, and we solve completely the associated Riemann problem.

A.1 Introduction

During the past fifty years, a wide range of models of vehicular traffic flow has been developed. Roughly speaking, three important classes of approaches are commonly used to model traffic phenomena. (i) *Microscopic models* or *Car-following models* e.g. [31], [4]: they are based on supposed mechanisms describing the process of one vehicle following another; (*ii*) *Kinetic models* e.g. [38], [60], [52], [43], [51], [40], [22]: they describe the dynamics of the velocity distribution of vehicles, in the traffic flow; (*iii*) *Fluid-dynamical models* e.g. [47], [62], [61], [18], [38], [3], [73], [33], [17], [6], [21]: they describe the dynamics of macroscopic variables (e.g. density, velocity, and flow) in space and time.

Here, we are concerned with the latter approch, i.e., the fluid-dynamical models. The first fluid model is due to M. J. Lighthill & G. B. Whitham, [47] and P. I. Richards, [62]. It consists of a single equation, the continuity one, thereby it is called "first order" model. Since then, various modifications and extensions to this basic model have been proposed in the literature. At the same time, nonequilibrium "second order" models, which consist of the continuity equation coupled with another equation describing the acceleration behaviour, have been developed. They are based either on perturbations of the isentropic gas dynamics models, see e.g. [61], [42] and [36], or on heuristic considerations and a derivation from the Follow-the-Leader model (FLM) see e.g. [3], [6], [33], [17] and [73].

In this paper, we propose a second order model, called the *Second Order Model with Constraints* (SOMC), which we derived from the A. Aw & M. Rascle (AR) model, [3] through a singular limit. We prove an existence result of weak solutions for such a model and discuss the associated Riemann problem. In contrast with the model introduced in [6] which assumes that the maximal density is constant (therefore independent of the velocity), here, we take into account the dependence of the maximal density constraint on the velocity. This consideration leads to a more realistic formulation, since it is well known that in practice, the distribution of vehicles on a highway, depends on their velocity. Furthermore, the particularity of the model we propose here, is its double-sided behaviour. Indeed, when the density constraint is saturated i.e., the maximal density is attained, for a given velocity, the SOMC model behaves like the Lighthill & Whitham first order model, whereas in the free flow regime, our model behaves like the pressureless gas model. Moreover, even in the Riemann problem, the interaction between two constant states in either regime can produce new states in the other regime: in other words the two regimes are intimately coupled and thus cannot ignore each other. Due to this specific property, we expect our model to capture some traffic complex phenomena such as stop and go waves.

The remaining parts of the paper are organized as follows. In Section A.2, we first introduce the SOMC model, justify its motivations, then we outline and discuss sufficient conditions for its derivation from the A. Aw & M. Rascle second order model, [3]. Section A.3 provides an existence result of weak solutions to the SOMC system. The Riemann Problem for the SOMC model is completely discussed in Section A.4. We finally conclude with directions for further research in Section A.5.

A.2 The model and its derivation

In this section, we present the Second Order Model with Constraints and highlight its specific properties. We justify the motivations of this model and discuss its derivation from the A. Aw & M. Rascle second order model, [3].

A.2.1 The Second Order Model with Constraints (SOMC)

The second order model we introduce in this paper, namely the *Second Order Model with Constaint* (in short SOMC) describes two different traffic regimes: the free flow regime, in which the vehicles are going with their preferred velocity and the congested regime where the velocity of vehicles is layed down by the traffic condition. The original feature of this model is its double-sided behaviour between two hyperbolic models, through an a priori unknown interface: in the free flow regime the model behaves like the pressureless gas model, whereas in the congested regime i.e., when the maximal density is attained for a given velocity, the SOMC model behaves like the Lighthill & Whitham first order model.

A.2.2 Derivation of the SOMC model

This paragraph is dedicated to the derivation of the SOMC model from the A. Aw & M. Rascle second order model, [3]. For sake of completeness, we present first the classical case in which the maximal density n^* is constant (i.e., independent of the velocity). Then, we introduce the case $n^* := n^*(u)$ and justify its motivations. Afterwards we discuss the derivation of the SOMC model from the A. Aw & M. Rascle model through a singular limit.

The case n^*=constant

In conservative form, the A. Aw & M. Rascle (AR) macroscopic model, [3] consists of the following equations:

$$\partial_t n + \partial_x(nu) = 0, \tag{A.1}$$
$$\partial_t(nw) + \partial_x(nwu) = 0, \tag{A.2}$$
$$w = u + p(n), \tag{A.3}$$

where $n(x,t)(\geqslant 0)$ and $u(x,t)(\geqslant 0)$ denote respectively the local density (number of vehicles per unit of space) and the velocity, both at the position x and the time t. The variable w denotes the drivers "preferred velocity" and $0 \leqslant p(n) \leqslant \infty$ is the velocity offset between the actual velocity and the preferred velocity.
In what follows we give some important properties of the AR model and refer the reader to [3] for more details.
Let us rewrite the system (A.1)-(A.3) in the following general form

$$\partial_t U + A(U)\partial_x U = 0 \tag{A.4}$$

$$\text{with } U = \begin{pmatrix} n \\ u \end{pmatrix} \text{ and } A(U) = \begin{pmatrix} u & n \\ 0 & u - np'(n) \end{pmatrix}. \tag{A.5}$$

The system (A.4)-(A.5) (or (A.1)-(A.3)) is strictly hyperbolic away from the vacuum (i.e. when $n \neq 0$). Indeed, the eigenvalues of the jacobian matrix $A(U)$ are

$$\lambda_1 = u - np'(n) \quad \leqslant \quad \lambda_2 = u. \tag{A.6}$$

and the associated eigeinvectors are respectively

$$r_1 = \begin{pmatrix} 1 \\ -p'(n) \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalues of the system correspond to the information propagation speed and they are both bounded by the traffic flow speed. Thereby the model complies with the anisotropic features of traffic flow.
Since $\nabla\lambda_1 \cdot r_1 \neq 0$ and $\nabla\lambda_2 \cdot r_2 = 0$ (here $\nabla := (\partial_n, \partial_{nw})$), then λ_1 is genuinely nonlinear and λ_2 is linearly degenerate. Therefore, the waves associated to λ_1 correspond to shock waves (braking) or rarefaction waves (acceleration) which modify the velocity, whereas the waves associated to λ_2 correspond to contact discontinuities. In this model, the shock and rarefaction curves coincide, therefore the model

falls into the class of "Temple Systems", [68]. The Riemann invariants in the sense of Lax, [45] for the system (A.1)-(A.3) are respectively w and u.

Naturally in the traffic dynamics, at each time $t > 0$, the following constraints have to be satisfied

$$0 \leqslant u(.,t) \leqslant u^*, \tag{A.7}$$

$$0 \leqslant n(.,t) \leqslant n^*, \tag{A.8}$$

with $u^* < \infty$ and $n^* < \infty$ respectively the maximal velocity and density.
In the (u, w) plane, this region is defined by

$$R_{u^*,n^*} = \{0 \leqslant u \leqslant u^*, \quad 0 \leqslant w - u \leqslant p(n^*)\}, \tag{A.9}$$

which is not an invariant region for the AR model. Therefore, for some badly chosen initial data in R_{u^*,n^*}, one may obtain for some (x, t), solutions which are, later on, out of the region R_{u^*,n^*}. There are two possible strategies to avoid the possible (unpleasant!) appearance of densities $n > n^*$ in the future. One consists in using invariant rectangles in the plane (u, w), see [3]. The other one is to choose a velocity offset p which is singular at $n = n^*$. One of the good candidate proposed in [6] is

$$p(n) = \left(\frac{1}{n} - \frac{1}{n^*}\right)^{-\gamma} \quad \text{with} \quad n \leqslant n^* \quad \text{and} \quad \gamma > 0, \tag{A.10}$$

where n^* denotes the maximal density. The AR model with the constraints (A.7)-(A.8) and the function p given by (A.10), is called the Modified AR model (MAR). Obviously, the MAR model inherits the properties of the AR model, stated above (see [6]).

It is known that drivers do not reduce significantly their speed unless they are too closed to the maximal density, what means the velocity offset $p \longrightarrow 0$ in free flow traffic. In the MAR model, this can be taken into account by replacing the functional p by the rescaled one: εp, with $\varepsilon \longrightarrow 0$. Therefore the rescaled model can be stated as follows:

$$\partial_t n^\varepsilon + \partial_x(n^\varepsilon u^\varepsilon) = 0, \tag{A.11}$$

$$(\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(n^\varepsilon)) = 0, \tag{A.12}$$

where $p(n)$ is defined in (A.10). The system (A.11)-(A.12) is called the Rescaled Modified A. Aw & M. Rascle model (RMAR). Furthermore, it conserves the properties of the MAR model.

Now we recall briefly the Constrained Pressureless Gas Dynamics model and we refer the reader to [6] for more details on this model.
Due to the form of the modified velocity offset (A.10),

$$p(n) \longrightarrow \infty \quad \text{when} \quad n \longrightarrow n^*.$$

Assume that $\varepsilon p(n^\varepsilon)(x,t)$ has a limit: $\bar{p}(x,t) := \lim_{\varepsilon \longrightarrow 0} \varepsilon p(n^\varepsilon)(x,t)$. If $n = n^*$ at the point (x,t), $\bar{p}$ may become non zero and finite, and $\bar{p}$ turns out to be a Lagrangian multiplier of the constraint $n \leqslant n^*$. In others words,

$$\bar{p} = \begin{cases} 0 \quad \text{if} \quad n < n^*, \\ c \quad (0 < c < \infty) \quad \text{if} \quad n = n^*. \end{cases} \tag{A.13}$$

Finally, the formal limit of the RMAR system (A.11)-(A.12) leads to the Constrained Pressureless Gas Dynamics model (CPGD):

$$\partial_t n + \partial_x(nu) = 0, \tag{A.14}$$

$$(\partial_t + u\partial_x)(u + \bar{p}) = 0, \tag{A.15}$$

$$0 \leqslant n \leqslant n^*, \quad \bar{p} \geqslant 0, \quad (n^* - n)\bar{p} = 0. \tag{A.16}$$

The case $n^* = n^*(u)$

As in practice, the minimal distance between a driver and its leading car is an increasing function of the velocity, a more realistic formulation of traffic flow must include this fact. With this consideration, the velocity offset p takes the form

$$p(n,u) = \left(\frac{1}{n} - \frac{1}{n^*(u)}\right)^{-\gamma}, \quad \text{with} \quad n \leqslant n^*(u), \quad \text{and} \quad \gamma > 0. \tag{A.17}$$

With the functional p in the above form (A.17) the MAR model turns to

$$\partial_t n + \partial_x(nu) = 0, \tag{A.18}$$

$$(\partial_t + u\partial_x)(u + p(n,u)) = 0. \tag{A.19}$$

From now on, p and n^* denote respectively $p(n,u)$ and $n^*(u)$.
Now,

$$
\begin{aligned}
0 &= (\partial_t + u\partial_x)(u + p(n,u)) \\
&= \partial_t u + u\partial_x u + \partial_n p \partial_t n + \partial_u p \partial_t u + u\partial_n p \partial_x n + u\partial_u p \partial_x u \\
&= \partial_t u + u\partial_x u - n\partial_n p \partial_x u + \partial_u p \partial_t u + u\partial_u p \partial_x u \\
&= (1+\partial_u p)\partial_t u + [u(1+\partial_u p) - n\partial_n p]\partial_x u,
\end{aligned}
$$

then the system (A.18)-(A.19), called the Modified AR* model (MAR*), can be rewritten as

$$\partial_t U + A(U)\partial_x U = 0, \tag{A.21}$$

$$\text{with } U = \begin{pmatrix} n \\ u \end{pmatrix} \text{ and } A(U) = \begin{pmatrix} u & n \\ 0 & u - \frac{n\partial_n p}{1+\partial_u p} \end{pmatrix}. \tag{A.22}$$

The eigenvalues of the matrix $A(U)$ are

$$\lambda_1 = u - \frac{n\partial_n p}{1+\partial_u p} \leqslant \lambda_2 = u, \tag{A.23}$$

and the associated eigenvectors are respectively

$$r_1 = \begin{pmatrix} 1+\partial_u p \\ -\partial_n p \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $\nabla\lambda_2 \cdot r_2 = 0$, the second eigenvalue is linearly degenerate (here, $\nabla := (\partial_n, \partial_u)$), the waves associated to λ_2 are contact discontinuities.
Now let us consider the first eigenvalue λ_1. We have

$$\nabla\lambda_1 = \begin{pmatrix} -\frac{\partial_n p}{1+\partial_u p} - \frac{n\partial_{nn} p}{1+\partial_u p} + \frac{n\partial_n p \partial_{un} p}{(1+\partial_u p)^2} \\ 1 - \frac{n\partial_{un} p}{1+\partial_u p} + \frac{n\partial_n p \partial_{uu} p}{(1+\partial_u p)^2} \end{pmatrix},$$

then

$$\nabla\lambda_1 \cdot r_1 = -2\partial_n p + \frac{2n\partial_n p \partial_{un} p}{1+\partial_u p} - n\partial_{nn} p - \frac{n(\partial_n p)^2 \partial_{uu} p}{(1+\partial_u p)^2}. \tag{A.24}$$

Clearly, $\exists\,(n,u) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ such that $\nabla\lambda_1 \cdot r_1 \neq 0$, hence λ_1 is not linearly degenerate. Therefore we would like λ_1 to be genuinely nonlinear i.e. $\nabla\lambda_1 \cdot r_1 \neq 0$, $\forall\,(n,u) \neq (0,0)$. In fact, one can easily notice that we need some assumptions on the functional $n^* : u \longmapsto n^*(u)$.
We consider the following assumptions:

(A-1) $n^*(u)$ is twice continuously differentiable;

(A-2) $n^*(u)$ is strictly decreasing;

(A-3) $n^*(u)$ concave $\left(\dfrac{\mathrm{d}^2}{\mathrm{d}u^2}(n^*(u)) \leqslant 0\right)$.

The second assumption is quite natural, since the faster the vehicles the larger the spacing between them.

Lemma A.1. *Under assumptions **(A-1)-(A-3)**, the eigenvalue λ_1 is genuinely non linear.*

For readability reasons, the proof of this lemma is postponed in the Appendix at the end of the paper.

Since λ_1 is genuinely non linear, therefore, the associated waves are either shocks or rarefaction waves. The Riemann invariants in the sense of Lax, [45] associated to the eigenvalues λ_1 and λ_2 are respectively

$$w = u + p(n,u) \quad \text{and} \quad z = u. \tag{A.25}$$

For the same reason as in the previous paragraph, let us multiply by ε the velocity offset p in the model (A.18)-(A.19). Then, we obtain

$$\partial_t U^\varepsilon + A(U^\varepsilon)\partial_x U^\varepsilon = 0, \tag{A.26}$$

$$\text{with } U^\varepsilon = \begin{pmatrix} n^\varepsilon \\ \\ u^\varepsilon \end{pmatrix} \text{ and } A(U^\varepsilon) = \begin{pmatrix} u^\varepsilon & n^\varepsilon \\ \\ 0 & u^\varepsilon - \frac{\varepsilon n^\varepsilon \partial_n p}{1+\varepsilon\partial_u p} \end{pmatrix}. \tag{A.27}$$

Hence the eigenvalues and the Riemann invariants in the sense of Lax, [45] are respectively

$$\lambda_1^\varepsilon = u^\varepsilon - \frac{\varepsilon n^\varepsilon \partial_n p}{1+\varepsilon \partial_u p} \leqslant \lambda_2^\varepsilon = u^\varepsilon \tag{A.28}$$

and $w^\varepsilon = u^\varepsilon + \varepsilon p(n^\varepsilon, u^\varepsilon)$, $z^\varepsilon = u^\varepsilon$. This modification conserves the properties of the model (A.18)-(A.19). The system (A.26)-(A.27) is called the Rescaled Modified AR* Model (RMAR*)

Let $\bar{p} = \lim\limits_{\varepsilon \longrightarrow 0} \varepsilon p(n^\varepsilon, u^\varepsilon)(x,t)$ and $(n^\varepsilon, u^\varepsilon) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} (n,u)$, then the formal limit of the RMAR* model (A.26)-(A.27) is given by

$$\partial_t n + \partial_x(nu) = 0, \tag{A.29}$$
$$(\partial_t + u\partial_x)(u+\bar{p}) = 0, \tag{A.30}$$
$$0 \leqslant n(x,t) \leqslant n^*(u(x,t)), \quad \bar{p} \geqslant 0, \quad (n^*(u)-n)\bar{p} = 0, \tag{A.31}$$

which we call the Second Order Model with Constraints (SOMC).

Proposition A.2. *Assume that* $\varepsilon p(n^\varepsilon, u^\varepsilon) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} \bar{p} > 0$, $u^\varepsilon(x,t) \longrightarrow u$ *and* $n^\varepsilon - n^*(u^\varepsilon) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} 0$. *Then*

$$\lambda_1^\varepsilon = u^\varepsilon - \frac{\varepsilon n^\varepsilon \partial_n p}{1+\varepsilon \partial_u p} \underset{\varepsilon \longrightarrow 0}{\longrightarrow} u + \frac{n^*(u)}{(n^*)'(u)} = \bar{\lambda}_1(u) \tag{A.32}$$

Proof. Since $\underset{\varepsilon \longrightarrow 0}{\varepsilon p(n^\varepsilon, u^\varepsilon)} = \bar{p} > 0$, then

$$\exists\, \delta > 0 \;\text{ such that }\; \forall\, \varepsilon > 0,\; \varepsilon p(n^\varepsilon, u^\varepsilon) \geqslant \delta.$$

Thus we have

$$\varepsilon \partial_u p(n^\varepsilon, u^\varepsilon) = -\gamma \frac{(n^*)'(u^\varepsilon)}{n^*(u^\varepsilon)^2} \left(\frac{1}{n^\varepsilon} - \frac{1}{n^*(u^\varepsilon)} \right)^{-1} \varepsilon p(n^\varepsilon, u^\varepsilon) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} \infty.$$

Therefore,

$$\frac{\varepsilon n^\varepsilon \partial_n p}{1+\varepsilon \partial_u p} \underset{\varepsilon \longrightarrow 0}{\sim} \frac{n^\varepsilon \partial_n p}{\partial_u p} = -\frac{\frac{1}{n^\varepsilon}}{\frac{(n^*)'(u^\varepsilon)}{n^*(u^\varepsilon)^2}} \underset{\varepsilon \longrightarrow 0}{\longrightarrow} -\frac{n^*(u)}{(n^*)'(u)}.$$

Finally

$$\lambda_1^\varepsilon \underset{\varepsilon \longrightarrow 0}{\longrightarrow} u + \frac{n^*(u)}{(n^*)'(u)} = \bar{\lambda}_1(u).$$

$\square$

Moreover, we have

$$\frac{\mathrm{d}\bar{\lambda}_1(u)}{\mathrm{d}u} = 1 + \frac{(n^*)'(u)^2 - n^*(u)(n^*)''(u)}{(n^*)'(u)^2}.$$

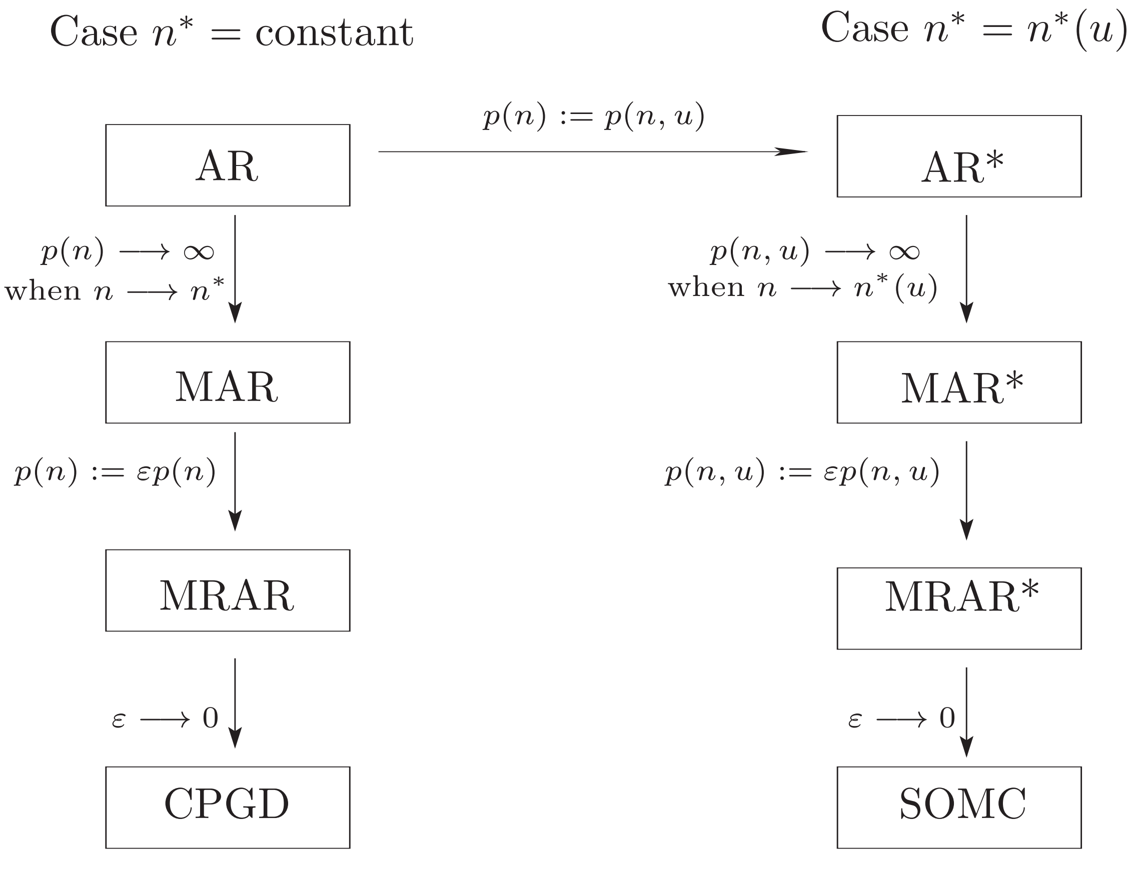


Figure A.1: Links between the different models.

Definition A.1. *We call a* ***cluster*** *or a* ***block****, a stretch of road defined by an interval* $[x_1(t), x_2(t)]$*, inside which the system (A.29)-(A.31) is satisfied and*

$$n(x,t) = \begin{cases} n^*(u(x,t)), & \text{if } x \in [x_1(t), x_2(t)]; \\ \\ 0, & \text{if } x \in [x_1(t) - \varepsilon(t), x_1(t)[\cup]x_2(t), x_2(t) + \varepsilon(t)], \text{ for } \varepsilon(t) \text{ small.} \end{cases}$$

Since $n^*(u)$ is concave, then $\bar{\lambda}_1(u)$ is strictly increasing. The limit $\bar{\lambda}_1$ is the characteristic speed of the Lighthill & Whitham model when $n = n^*(u)$. Contrarily to the case $n^* = constant$, here $|\bar{\lambda}_1| < +\infty$. In other words, a velocity variation in front of a cluster propages with a **finite** speed (but not with an infinite speed as in [6]) through the whole cluster.

A.2.3 Properties of the SOMC model

It is well known that in traffic, the minimal distance between a driver and its leading car is an increasing function of the velocity. Therefore, in contrast with the model introduced in [6], here the maximal density n^* is a functional of the velocity u. However, this natural consideration imparts to the SOMC model a particular property: a double-sided behaviour. Indeed, when $n(x,t) = n^*(u(x,t))$, i.e. the maximal density constraint $n(x,t) \leqslant n^*(u(x,t))$ is saturated, a block of vehicles (or a cluster) forms. In a cluster, u and $\bar{p}$ are layed down by the first vehicle, and as long as the cluster is going freely, these variables remain constant, see Section A.3 and the discussions in Section A.4 below. Therefore, inside each cluster which is going freely, the SOMC model writes

$$\partial_t n^*(u) + \partial_x (n^*(u)u) = 0. \tag{A.33}$$

Let $n \longrightarrow u^*(n)$ be the inverse functional of $u \longrightarrow n^*(u)$. Therefore (A.33) rewrites

$$\partial_t n + \partial_x (n u^*(n)) = 0, \tag{A.34}$$

where $q(n) := nu^*(n)$ is the flux function as in the Lighthill & Whitham model, [47]. Therefore, we have a hyperbolic second order model which "relaxes" to the Lighthill & Whitham first order model when the maximal density constraint is saturated. Hence, the SOMC model is expected to capture the stop and go waves phenomena since there is no invariant region for the velocity u when the model behaves as the Lighthill & Whitham model.

A.3 Existence result for the SOMC model (A.29)-(A.31)

This section is devoted to the proof of the existence of weak solutions to the SOMC system (A.29)-(A.31), written in conservative form. The proof is based

on the results in [5] and is strongly motivated by the analysis of the Riemann problem in Section A.4. Indeed, this analysis permits us to exhibit the limit as $\varepsilon \longrightarrow 0$ of the solutions to the Riemann problem of the RMAR* model (A.26)-(A.27), which are nothing but the expected solutions to the Riemann problem of the SOMC model (A.29)-(A.31). For instance, when two blocks collide i.e., the cluster behind is going faster than the cluster ahead, a shock wave appears at the front of the cluster behind and propagates upstream with a finite speed. This technical and "self contained" analysis is postponed at the end of the paper for readability reasons. However, it is not needless since it justifies the choice of the dynamics considered below, and allows us to expect that the obtained solution (non unique) is the one which models the real phenomena. First we prove the existence of weak solutions for some particular data and then we prove the stability of the obtained solutions. Namely, we make use of the result in [5], in which it has been proved that any smooth function can be approximated in the distribution sense, by a sequence of characteristic functions.
In conservative form, the SOMC model (A.29)-(A.31) is written as follows:

$$\partial_t n + \partial_x(nu) = 0, \tag{A.35a}$$

$$\partial_t(n(u+\bar{p})) + \partial_x(n(u+\bar{p})u) = 0, \tag{A.35b}$$

$$0 \leqslant n(x,t) \leqslant n^*(u(x,t)), \quad \bar{p} \geqslant 0, \quad (n^*(u)-n)\bar{p} = 0. \tag{A.35c}$$

A.3.1 Clusters dynamics

In order to prove the existence of solutions for the SOMC model, we mimic the approach in [11] (which was also used in [6]). We approximate the initial datum as a succession of vacuum and blocks (or clusters) where the constraint is saturated. Physically, this means that any traffic condition can be approximated in the weak sense by a situation where saturated stretches of road are followed by empty stretches. So, our first task is to consider the dynamics of a solution which consists of a succession of clusters and vacuum. In particular, the key point in defining this dynamics is to specify what happens when a faster cluster meets a slower one in front. To define what happens when two clusters meet, we take inspiration from the examination of the solutions of the Riemann problem, which is developed in Section A.4. In what follows, we construct the cluster (or block) solutions to (A.35a)-(A.35c).
Now let us consider the density $n(x,t)$, the flux $n(x,t)u(x,t)$ and the quantity $n(x,t)\bar{p}(x,t)$ given respectively by

$$n(x,t) = \sum_{i=1}^{N} n_i^*(t)\,\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{A.36}$$

$$n(x,t)u(x,t) = \sum_{i=1}^{N} n_i^*(t)u_i(t)\,\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{A.37}$$

$$n(x,t)\bar{p}(x,t) = \sum_{i=1}^{N} n_i^*(t)\bar{p}_i(t)\,\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{A.38}$$

with $n_i^*(t) = n^*(u_i(t))$ (or equivalently $u_i = u^*(n_i)$) as long as there is no collision. That is to say $a_N(t) < b_N(t) < a_{N-1}(t) < b_{N-1}(t) < \ldots < b_1(t)$ and the number of blocks N is constant until there is a collision.

If there is no collision, each block i moves freely with a constant velocity, i.e. $u_i(t) := u_i$. Therefore $n_i^*(t) = n^*(u_i) := n_i^*$ is also constant. On the other hand, when a block $i+1$ catches up with the block ahead i at time t^* (this implies in particular that $u_{i+1} > u_i$), then a shock wave appears and propagates gradually inside the block $i+1$. The shock speed u_s is given by

$$u_s = \frac{n_i^* u_i - n_{i+1}^* u_{i+1}}{n_i^* - n_{i+1}^*}. \tag{A.39}$$

We notice that since $u_{i+1} > u_i$ then necessarily $n_i^* \neq n_{i+1}^*$.
The dynamics is illustrated by Figure A.2.

Let $\sigma(t)$ be the wave trajectory i.e. $\sigma'(t) = u_s$. In contrast to [6], here the block $(i+1)$ does not take instantaneously the velocity u_i but it adjusts its velocity gradually through $\sigma(t)$. Let t^{**} be the time at which $\sigma(t)$ reaches the left boundary of the block $i+1$, see Figure A.2. Around this shock, the density $n(x,t)$, the flux $n(x,t)u(x,t)$ and the functional $\bar{p}(x,t)$ are locally given respectively by

$$n(x,t) = \begin{cases} n_i^*\mathbb{1}_{a_i(t)<x<b_i(t)} + n_{i+1}^*\mathbb{1}_{a_{i+1}(t)<x<b_{i+1}(t)} : & \text{if } \ t < t^*, \\ n_i^*\mathbb{1}_{a_i(t)<x<b_i(t)} + n_i^*\mathbb{1}_{\sigma(t)<x<a_i(t)} & \\ +n_{i+1}^*\mathbb{1}_{a_{i+1}(t)<x<\sigma(t)} : & \text{if } \ t^* < t < t^{**}, \\ n_i^*\mathbb{1}_{a_i(t)<x<b_i(t)} + n_i^*\mathbb{1}_{\tilde{a}_i(t)<x<a_i(t)} : & \text{if } \ t > t^{**}, \end{cases} \tag{A.40}$$

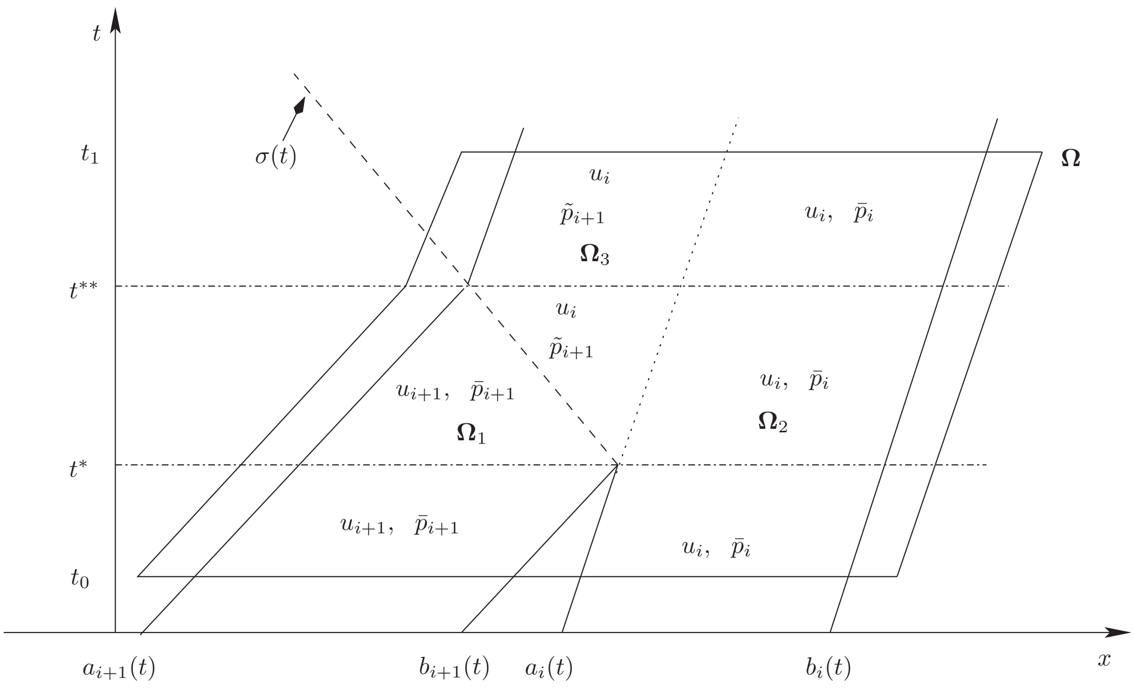


Figure A.2: The traffic dynamics when two blocks collide.

$$n(x,t)u(x,t) = \begin{cases} n_i^* u_i \mathbb{1}_{a_i(t)<x<b_i(t)} + n_{i+1}^* u_{i+1} \mathbb{1}_{a_{i+1}(t)<x<b_{i+1}(t)} : \text{if} \quad t < t^*, \\ n_i^* u_i \mathbb{1}_{a_i(t)<x<b_i(t)} + n_i^* u_i \mathbb{1}_{\sigma(t)<x<a_i(t)} \\ +n_{i+1}^* u_{i+1} \mathbb{1}_{a_{i+1}(t)<x<\sigma(t)} : \qquad \text{if} \quad t^* < t < t^{**}, \\ n_i^* u_i \mathbb{1}_{a_i(t)<x<b_i(t)} + n_i^* u_i \mathbb{1}_{\tilde{a}_i(t)<x<a_i(t)} : \qquad \text{if} \quad t > t^{**} \end{cases} \tag{A.41}$$

and

$$n(x,t)\bar{p}(x,t) = \begin{cases} n_i^* \bar{p}_i \mathbb{1}_{a_i(t)<x<b_i(t)} + n_{i+1}^* \bar{p}_{i+1} \mathbb{1}_{a_{i+1}(t)<x<b_{i+1}(t)} : \text{if} \quad t < t^*, \\ n_i^* \bar{p}_i \mathbb{1}_{a_i(t)<x<b_i(t)} + n_i^* \tilde{p}_{i+1} \mathbb{1}_{\sigma(t)<x<a_i(t)} \\ +n_{i+1}^* \bar{p}_{i+1} \mathbb{1}_{a_{i+1}(t)<x<\sigma(t)} : \qquad \text{if} \quad t^* < t < t^{**}, \\ n_i^* \bar{p}_i \mathbb{1}_{a_i(t)<x<b_i(t)} + n_i^* \tilde{p}_{i+1} \mathbb{1}_{\tilde{a}_i(t)<x<a_i(t)} : \qquad \text{if} \quad t > t^{**}, \end{cases} \tag{A.42}$$

with

$$\tilde{p}_{i+1} = \bar{p}_{i+1} + u_{i+1} - u_i \geqslant 0, \tag{A.43}$$

and

$$\tilde{a}_i' = u_i, \text{ and } \tilde{a}_i(t^{**}) = \sigma(t^{**}). \tag{A.44}$$

Remarks A.1. *The velocity u and the "pressure" $\bar{p}$ are assumed to be extended linearly in the vacuum ($n = 0$) between two successive blocks. Moreover we assume that u and $\bar{p}$ are constant at $\pm\infty$.*

A.3.2 Properties of the cluster dynamics

Let us start this section by the following result.

Theorem A.1. *With the above dynamics, the quantities $n(x,t)$, $u(x,t)$ and $\bar{p}(x,t)$ defined by (A.36)-(A.38) and Remark A.1 are solutions to (A.35a)-(A.35c).*

Proof. When there is no collision, each block i moves freely at a constant velocity $u_i(t) := u_i$. The density $n_i^*(t) = n^*(u_i) := n_i^*$ and the "pressure" $\bar{p}_i(t)$ are also constant in each block i. Then, $(n, u, \bar{p})$ defined by (A.36)-(A.37)-(A.38) solves the system (A.35a)-(A.35c). Now let us turn to the case of collision of two blocks at time t^* in the above dynamics. Let Ω be a domain which only contains the two blocks concerned with this collision, see Figure A.2. Then Ω is given by

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3, \tag{A.45}$$

where

$$\begin{aligned}
\Omega_1 =& \{(x,t);\ a_{i+1}(t) \leqslant x \leqslant b_{i+1}(t) \text{and } t_0 < t \leqslant t^*\} \\
& \cup \{(x,t);\ a_{i+1}(t) \leqslant x \leqslant \sigma(t) \text{ and } t^* \leqslant t \leqslant t^{**}\} \\
\Omega_2 =& \{(x,t);\ a_i(t) \leqslant x \leqslant b_i(t) \text{ and } t_0 < t \leqslant t^*\} \\
& \cup \{(x,t);\ a_i(t) \leqslant x \leqslant b_i(t) \text{ and } t^* \leqslant t \leqslant t^{**}\} \\
& \cup \{(x,t);\ a_i(t) \leqslant x \leqslant b_i(t) \text{ and } t^{**} \leqslant t \leqslant t_1\} \\
\Omega_3 =& \{(x,t);\ \sigma(t) \leqslant x \leqslant a_i(t) \text{ and } t^* \leqslant t \leqslant t^{**}\} \\
& \cup \{(x,t);\ \tilde{a}_i(t) \leqslant x \leqslant a_i(t) \text{ and } t^{**} \leqslant t \leqslant t_1\}, (\text{ with } \tilde{a}_i(t) \equiv a_{i+1}(t)).
\end{aligned}$$

Let $\varphi(x,t)$ be a smooth function with compact support in Ω. For any continuous function S and denoting by $<,>$ the distribution duality brackets, we have

$$\begin{aligned}
< \partial_t(nS(u,\bar{p})) + \partial_x(nuS(u,\bar{p})), \varphi > &= -\iint_\Omega nS(u,\bar{p})(\partial_t\varphi + u\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&= A_1 + A_2 + A_3
\end{aligned}$$

where $A_j = \iint_{\Omega_j} nS(u,\bar{p})(\partial_t\varphi + u\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t$, for $j = 1\ldots 3$.
For all $i = 1\ldots N$, we have

$$\int_{a_i(t)}^{b_i(t)} \partial_x\varphi(x,t)\,\mathrm{d}x = \varphi(b_i(t),t) - \varphi(a_i(t),t). \tag{A.46}$$

On the other hand

$$\int_{a_i(t)}^{b_i(t)} \partial_t\varphi(x,t)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{a_i(t)}^{b_i(t)} \varphi(x,t)\,\mathrm{d}x\right] - \varphi(b_i(t),t)b_i'(t) + \varphi(a_i(t),t)a_i'(t).$$

Furthermore, for a given block i, we have $b_i'(t) = a_i'(t) = u_i$ and on the shock wave $\sigma' = u_s$. Therefore, since φ has a compact support Ω, we compute

$$\begin{aligned} A_1 = &\int_{a_{i+1}(t^{**})}^{\sigma(t^{**})} n_{i+1}^* S(u_{i+1},\bar{p}_{i+1})\varphi(x,t^{**})\,\mathrm{d}x \\ &+ \int_{\sigma(t^*)}^{b_{i+1}(t^*)} n_{i+1}^* S(u_{i+1},\bar{p}_{i+1})\varphi(x,t^*)\,\mathrm{d}x \\ &+ \int_{t^*}^{t^{**}} n_{i+1}^* S(u_{i+1},\bar{p}_{i+1})(u_{i+1}-u_s)\varphi(\sigma(t),t)\,\mathrm{d}t; \end{aligned} \tag{A.47}$$

$$A_2 = 0; \tag{A.48}$$

$$\begin{aligned} A_3 = &\int_{a_i(t^*)}^{\sigma(t^*)} n_i^* S(u_i,\tilde{p}_{i+1})\varphi(x,t^*)\,\mathrm{d}x \\ &+ \int_{\sigma(t^{**})}^{\tilde{a}_i(t^{**})} n_i^* S(u_i,\tilde{p}_{i+1})\varphi(x,t^{**})\,\mathrm{d}x \\ &+ \int_{t^*}^{t^{**}} n_i^* S(u_i,\tilde{p}_{i+1})(u_s-u_i)\varphi(\sigma(t),t)\,\mathrm{d}t. \end{aligned} \tag{A.49}$$

Now,

$$a_i(t^*) = b_{i+1}(t^*) = \sigma(t^*), \sigma(t^{**}) = \tilde{a}_i(t^{**}), a_{i+1}(t^{**}) = \sigma(t^{**}), \tag{A.50}$$

then

$$\begin{aligned} &< \partial_t(nS(u,\bar{p})) + \partial_x(nuS(u,\bar{p})), \varphi > \\ =&[n_i^*(u_s-u_i)S(u_i,\tilde{p}_{i+1}) - n_{i+1}^*(u_s-u_{i+1})S(u_{i+1},\bar{p}_{i+1})] \int_{t^*}^{t^{**}} \varphi(\sigma(t),t)\,\mathrm{d}t. \end{aligned} \tag{A.51}$$

For $S(u, \bar{p}) = 1$, (A.51) turns to

$$\begin{aligned} &< \partial_t n, \varphi > + < \partial_x (nu), \varphi > \\ =&[n_i^*(u_s - u_i) - n_{i+1}^*(u_s - u_{i+1})] \int_{t^*}^{t^{**}} \varphi(\sigma(t), t)\,\mathrm{d}t. \end{aligned} \tag{A.52}$$

From (A.39), we have

$$n_i^*(u_i - u_s) = n_{i+1}^*(u_{i+1} - u_s), \tag{A.53}$$

therefore

$$< \partial_t n, \varphi > + < \partial_x (nu), \varphi >= 0. \tag{A.54}$$

For $S(u, \bar{p}) = u + \bar{p}$, we obtain:

$$\begin{aligned} &< \partial_t (n(u + \bar{p})), \varphi > + < \partial_x (n(u + \bar{p})u), \varphi > \\ =&[(u_i + \tilde{p}_{i+1}) - (u_{i+1} + \bar{p}_{i+1})] \times n_i^*(u_s - u_i) \int_{t^*}^{t^{**}} \varphi(\sigma(t), t)\,\mathrm{d}t, \end{aligned} \tag{A.55}$$

which implies, thanks to (A.39) and (A.43)

$$< \partial_t (n(u + \bar{p})), \varphi > + < \partial_x (n(u + \bar{p})u) >= 0. \tag{A.56}$$

□

Proposition A.3. *We have the maximum principle*

$$\operatorname*{essinf}_y u^0(y) \leqslant u(x,t) \leqslant \operatorname*{esssup}_y u^0(y), \tag{A.57}$$

where esssup *and* essinf *denote respectively the essential* sup *and the essential* inf*. We also have the bound*

$$0 \leqslant \bar{p}(x,t) \leqslant \operatorname*{esssup}_y u^0(y) + \operatorname*{esssup}_y \bar{p}^0(y). \tag{A.58}$$

Assume furthermore that the initial data in the blocks u_i^0 and $\bar{p}_i^0$ are BV functions. Then we have for all $t \in [0, T]$

$$TV_K(u(.,t)) \leqslant TV_{\tilde{K}}(u^0), \tag{A.59}$$

$$TV_K(\bar{p}(.,t)) \leqslant TV_{\tilde{K}}(\bar{p}^0) + 2TV_{\tilde{K}}(u^0), \tag{A.60}$$

for any compact $K = [a, b]$ and with $\tilde{K} = [a - t\,\mathrm{esssup}\,|u^0|\,, b - t\,\mathrm{essinf}\,|u^0|]$ where TV_K (resp. $TV_{\tilde{K}}$) denotes the total variation on the set K (resp. $\tilde{K}$).

Proof. We gave below the proof of (A.60) for some extreme cases but the proof is general. For all $i = 1 \dots N$, when the block $i+1$ collides with the block i then, $\bar{p}_{i+1}$ becomes $\tilde{p}_{i+1} = \bar{p}_{i+1} + u_{i+1} - u_i$. We assume the following dynamics: in a time interval $[0, t] \subset [0, T]$, j blocks collide successively at $t_1, \cdots, t_{j-1} \leqslant t$ for instance and then $N - j + 1$ blocks collide at the same time $t_s \leqslant t$, then we have

$$
\begin{aligned}
TV_K(\bar{p}(t,.)) = & \left|\bar{p}_1^0 - \tilde{p}_2^{t_1}\right| + \left|\tilde{p}_2^{t_1} - \tilde{p}_3^{t_2}\right| + \ldots + \left|\tilde{p}_{j-1}^{t_{j-2}} - \tilde{p}_j^{t_{j-1}}\right| \\
& + \left|\tilde{p}_j^{t_s} - \tilde{p}_{j+1}^{t_s}\right| + \ldots + \left|\tilde{p}_{N-1}^{t_s} - \tilde{p}_N^{t_s}\right| \\
& \leqslant |\bar{p}_1^0 - \bar{p}_2^0| + |\bar{p}_2^0 - \bar{p}_3^0| + \ldots + \left|\bar{p}_{j-1}^0 - \bar{p}_j^0\right| + \\
& 2\left(|u_1^0 - u_2^0| + |u_2^0 - u_3^0| + \ldots + \left|u_{j-1}^0 - u_j^0\right|\right) \\
& + \left|\bar{p}_j^{t_{s-1}} - \bar{p}_{j+1}^{t_{s-1}}\right| + \ldots + \left|\bar{p}_{N-1}^{t_{s-1}} - \bar{p}_N^{t_{s-1}}\right| + \\
& 2\left(\left|u_j^{t_{s-1}} - u_{j+1}^{t_{s-1}}\right| + \ldots + \left|u_{N-1}^{t_{s-1}} - u_N^{t_{s-1}}\right|\right) \\
& \leqslant TV_{\tilde{K}}(\bar{p}^0) + 2TV_{\tilde{K}}(u^0).
\end{aligned}
$$

□

A.3.3 Existence of a weak solution

In the previous section we have proved the existence of solution to (A.35a)-(A.35c) for some particular data. We prove now that these particular initial data are dense, in some sense, in the set of desired initial data.

Lemma A.4. *Let $n^0 \in L^1(\mathbb{R})$, $u^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ such that $0 \leqslant n^0 \leqslant n^*(u^0)$, then there exists a sequence of block initial data $(n_k^0, u_k^0, 0)_{k\geqslant 0}$ such that $\int_{\mathbb{R}} n_k^0(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} n^0(x)\,\mathrm{d}x$, $\operatorname{essinf} u^0 \leqslant u_k^0 \leqslant \operatorname{esssup} u^0$ and $TV(u_k^0) \leqslant TV(u^0)$ for which the convergences $n_k^0 \underset{k\to\infty}{\rightharpoonup} n^0$ and $n_k^0 u_k^0 \underset{k\to\infty}{\rightharpoonup} n^0 u^0$ hold in the distribution sense.*

Proof. The proof is widely inspired from the one of Lemma 4.1 in [5]. Let $k \in \mathbb{N}^*$ and let set $\forall\ i \in \mathbb{Z}$

$$
m_{ik} = \int_{\frac{i}{k}}^{\frac{i+1}{k} - \frac{1}{k^2}} n^0(x)\,\mathrm{d}x.
$$

If $m_{ik} \neq 0$, we set

$$\begin{aligned}
u_{ik}^0 &= \inf_{[\frac{i}{k}, \frac{i+1}{k}]} u^0(x)\,\mathrm{d}x, \\
n_{ik}^0 &= n^*(u_{ik}^0).
\end{aligned}$$

We note that, since $n^0(x) \leqslant n^*(u^0(x)) \leqslant n^*(u_{ik}^0) = n_{ik}^0$, then we have $\frac{m_{ik}}{n_{ik}^0} < \frac{1}{k}$. We finally set, for any $x \in \mathbb{R}$,

$$n_k^0(x) = \sum_{i=-k^2}^{k^2} n_{ik}^0 \mathbb{1}_{\left]\frac{i}{k}, \frac{i}{k} + \frac{m_{ik}}{n_{ik}^0}\right[}(x), \tag{A.61}$$

$$n_k^0(x) u_k^0(x) = \sum_{i=-k^2}^{k^2} n_{ik}^0 u_{ik}^0 \mathbb{1}_{\left]\frac{i}{k}, \frac{i}{k} + \frac{m_{ik}}{n_{ik}^0}\right[}(x). \tag{A.62}$$

We extend the definition of u_k^0 in the vacuum as in Remark A.1. We notice that we have

$$TV_{[a,b]}(u_k^0) \leqslant TV_{[a-1/k, b+1/k]}(u^0). \tag{A.63}$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$ and let $k_0 \in \mathbb{N}$ such that *supp* $\varphi \subset [-k_0, k_0]$. We have

$$\begin{aligned}
\int_{\mathbb{R}} n_k^0(x)\varphi(x)\,\mathrm{d}x &= \sum_{i=-k^2}^{k^2} \int_{\frac{i}{k}}^{\frac{i}{k}+\frac{m_{ik}}{n_{ik}^0}} n_{ik}^0 \varphi(x)\,\mathrm{d}x \\
&= \sum_{i=-k^2}^{k^2} \left[\varphi\left(\frac{i}{k}\right) m_{ik} + \varphi'(x_k^i)\frac{m_{ik}^2}{2n_{ik}^0}\right] \\
&= \sum_{i=-k^2}^{k^2} \left[\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} n^0(x)\varphi\left(\frac{i}{k}\right)\mathrm{d}x + \varphi'(x_k^i)\frac{m_{i,k}^2}{2n_{ik}^0}\right]
\end{aligned}$$

where $x_k^i \in \left[\frac{i}{k}, \frac{i}{k} + \frac{m_{ik}}{n_{ik}^0}\right]$.

For $k > k_0$, we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} n_k^0(x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}} n^0(x)\varphi(x)\,\mathrm{d}x \right| \\
\leqslant\ & \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \left\|n^0\right\|_\infty \left|\varphi\left(\frac{i}{k}\right) - \varphi(x)\right| \mathrm{d}x + \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i+1}{k}-\frac{1}{k^2}}^{\frac{i+1}{k}} \left\|n^0\right\|_\infty |\varphi(x)|\,\mathrm{d}x \\
&+ \sum_{i=-kk_0}^{kk_0-1} \left|\varphi'(x_k^i)\right| \frac{m_{i,k}^2}{2n_{ik}^0} \\
\leqslant\ & \left\|n^0\right\|_\infty \|\varphi'\|_\infty \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \left(x - \frac{i}{k}\right) \mathrm{d}x + \left\|n^0\right\|_\infty \|\varphi\|_\infty \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i+1}{k}-\frac{1}{k^2}}^{\frac{i+1}{k}} \mathrm{d}x \\
&+ \|\varphi'\|_\infty \sum_{i=-kk_0}^{kk_0-1} \left\|n^0\right\|_\infty \left(\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \mathrm{d}x \right)^2 \\
\leqslant\ & \frac{k_0}{k} \left\|n^0\right\|_\infty \|\varphi'\|_\infty + \frac{2k_0}{k} \left\|n^0\right\|_\infty \|\varphi\|_\infty + \frac{2k_0}{k} \left\|n^0\right\|_\infty \|\varphi'\|_\infty = O\left(\frac{1}{k}\right)
\end{aligned}
$$

and then

$$
\left\langle n_k^0, \varphi \right\rangle \underset{k \longrightarrow \infty}{\longrightarrow} \left\langle n^0, \varphi \right\rangle. \tag{A.64}
$$

Similarly, we have

$$
\int_{\mathbb{R}} n_k^0(x) u_k^0(x) \varphi(x)\,\mathrm{d}x = \sum_{-k^2}^{k^2} \left[\int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} n^0(x) u_{ik}^0 \varphi\left(\frac{i}{k}\right) \mathrm{d}x + u_{ik}^0 \varphi'(x_k^i) \frac{m_{ik}^2}{2n_{ik}^0} \right],
$$

and the main difference to prove the convergence

$$
\left\langle n_k^0 u_k^0, \varphi \right\rangle \underset{k \longrightarrow \infty}{\longrightarrow} \left\langle n^0 u^0, \varphi \right\rangle, \tag{A.65}
$$

is to show that

$$
A = \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \left|n^0(x)\right| \left|u_{ik}^0 - u^0(x)\right| \left|\varphi\left(\frac{i}{k}\right)\right| \mathrm{d}x \underset{k \longrightarrow \infty}{\longrightarrow} 0.
$$

This last fact comes from the majoration

$$
\begin{aligned}
A &\leqslant \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \left\|n^0\right\|_\infty \|\varphi\|_\infty \left|u^0_{ik}-u^0(x)\right| \mathrm{d}x \\
&\leqslant \left\|n^0\right\|_\infty \|\varphi\|_\infty \sum_{i=-kk_0}^{kk_0-1} \int_{\frac{i}{k}}^{\frac{i+1}{k}-\frac{1}{k^2}} \left| \sup_{\left[\frac{i}{k},\frac{i+1}{k}\right]} u^0 - \inf_{\left[\frac{i}{k},\frac{i+1}{k}\right]} u^0 \right| \mathrm{d}x \\
&\leqslant \frac{1}{k} \left\|n^0\right\|_\infty \|\varphi\|_\infty \sum_{i=-kk_0}^{kk_0-1} TV_x\left(u^0; \left[\frac{i}{k}, \frac{i+1}{k}\right]\right) \\
&\leqslant \frac{1}{k} \left\|n^0\right\|_\infty \|\varphi\|_\infty TV_x(u^0;[-k_0,k_0]).
\end{aligned}
$$

□

Contrarily to [6], in this paper, due to the finite wave speed, we have $u_k \longrightarrow u$ in L^1 (we will come back to this assertion in the proof of the Theorem A.2 below).

Lemma A.5. *, [5] Let us assume that $(\gamma_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\mathbb{R}\times]0,T[)$ that tends to γ in $L^\infty_{w^*}(\mathbb{R}\times]0,\infty[)$, and satisfies for any $\Gamma \in C^\infty_c(\mathbb{R})$,*

$$
\int_{\mathbb{R}} (\gamma_k-\gamma)(x,t)\Gamma(x)\,\mathrm{d}x \underset{k\longrightarrow\infty}{\longrightarrow} 0, \text{ in } L^1_t(]0,T[). \tag{A.66}
$$

Let us also assume that $(\omega_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\mathbb{R}\times]0,T[)$. If $\omega_k \longrightarrow \omega$ in L^1, then $\gamma_k\omega_k \rightharpoonup \gamma\omega$ in $L^\infty_{w^*}(\mathbb{R}\times]0,\infty[)$, as $k\longrightarrow\infty$

We are searching solutions with the following regularities

$$
\begin{aligned}
& n \in L^\infty_t(]0,\infty[\,, L^\infty_x(\mathbb{R})\cap L^1_x(\mathbb{R})), && \text{(A.67)}\\
& u,\bar{p} \in L^\infty_t(]0,\infty[\,, L^\infty_x(\mathbb{R})), && \text{(A.68)}\\
& 0 \leqslant n \leqslant n^*(u), \quad \bar{p}(n^*(u)-n)=0. && \text{(A.69)}
\end{aligned}
$$

and the existence result is:

Theorem A.2. *Let $(n^0,u^0,0)$ be some initial data such that*

$$
\begin{aligned}
& n^0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}), && \text{(A.70)}\\
& u^0 \in L^\infty(\mathbb{R})\cap BV(\mathbb{R}), && \text{(A.71)}\\
& \textit{with } 0\leqslant n^0 \leqslant n^*(u^0). && \text{(A.72)}
\end{aligned}
$$

Then there exists $(n, u, \bar{p})$ with regularities (A.67)-(A.69), solution to the system (A.35a)-(A.35c) with initial data $(n^0, u^0, 0)$. Moreover, this solution satisfies

$$\operatorname*{essinf}_y u^0(y) \leqslant u(x,t) \leqslant \operatorname*{esssup}_y u^0(y), \tag{A.73}$$

$$0 \leqslant \bar{p}(x,t) \leqslant \operatorname*{esssup}_y u^0(y). \tag{A.74}$$

Proof. Let n_k^0, u_k^0 and $\bar{p}_k^0 = 0$ $(k \in \mathbb{N})$ be the blocks initial data associated respectively to n^0, u^0 and $p^0 = 0$ provided by Lemma A.4. For all k, the results of Section A.3.2 allow us to get $(n_k, u_k, \bar{p}_k)$, solution of (A.35a)-(A.35c) with initial data (n_k^0, u_k^0, p_k^0), satisfying (A.67)-(A.69). We are going to use the compactness result in Lemma A.5 to prove that, up to a subsequence, as $k \longrightarrow \infty$, $(n_k, u_k, \bar{p}_k) \rightharpoonup (n, u, \bar{p})$, where $(n, u, \bar{p})$, with regularities (A.67)-(A.69), is a solution to (A.35a)-(A.35c) for initial data $(n^0, u^0, \bar{p}^0)$.
Since (n_k) is bounded in L^∞, then there exists a subsequence such that

$$n_k \rightharpoonup n \ \text{ in } \ L^\infty_{w*}(\mathbb{R}\times]0,\infty[). \tag{A.75}$$

Thanks to (A.57) and the bounds on u_k^0 provided by Lemma A.4, the sequence $(u_k)_{k\geqslant 0}$ is uniformely bounded in $L^\infty(\mathbb{R}\times]0,\infty[)$ and similarly $(\bar{p}_k)$ is bounded in L^∞, then we can extract subsequences such that we also have

$$u_k \rightharpoonup u \ \text{ in } \ L^\infty_{w*}(\mathbb{R}\times]0,\infty[), \quad \bar{p}_k \rightharpoonup \bar{p} \ \text{ in } \ L^\infty_{w*}(\mathbb{R}\times]0,\infty[). \tag{A.76}$$

We want now to prove the passage to the limit in the equation.
First, we study an important new property of the model which is directly related to the finite speed of propagation and gives a strong compactness for the velocity. In order to get it, we study the variation with respect to t of the L^1_x norm of u_k. From the Figure A.3, we see that the worst case of evolution of this quantity is related to the computation of an area which is bounded by $\|u_k\|_\infty |t_2 - t_1|$ times the variation of u_k between two blocks. According to the definition of u_k on the vacuum, the sum of all this quantity makes appear $TV(u^0)$.

Finally, we get that

$$\int_{\mathbb{R}} |u_k(x,t_2) - u_k(x,t_1)| \,\mathrm{d}x \leqslant \|u_k\|_\infty |t_2 - t_1| \, TV(u^0) = C\,|t_2 - t_1|\,. \tag{A.77}$$

From the BV_x bound on u_k in (A.59), this equicontinuity with respect to t and a Cantor diagonal process argument implies

$$u_k \underset{k\to\infty}{\longrightarrow} u \ \text{ in } \ L^1(\mathbb{R}\times[0,T]). \tag{A.78}$$

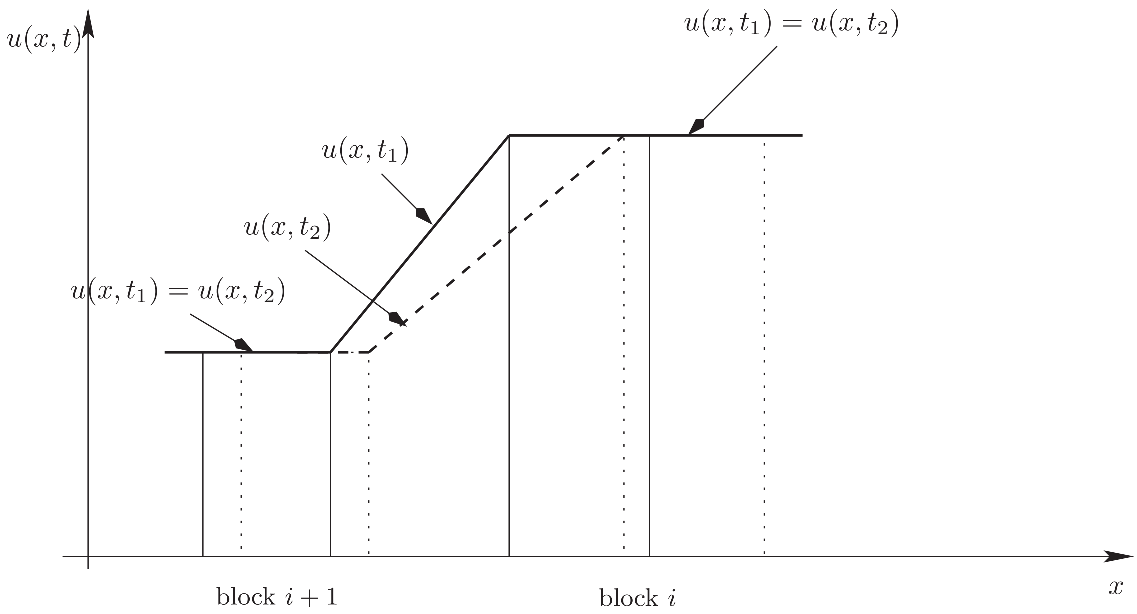


Figure A.3: The L^1 equicontinuity with respect to t .

Similarly, we also have

$$\bar{p}_k \underset{k\to\infty}{\longrightarrow} \bar{p} \quad \text{in} \quad L^1(\mathbb{R}\times[0,T]). \tag{A.79}$$

From the mass conservation equation, for any $\varphi \in C_c^\infty(\mathbb{R})$, the sequence

$$\int_{\mathbb{R}} n_k(t,x)\varphi(x)\,\mathrm{d}x$$

is bounded in BV_t. Then by Lemma A.5, we have

$$n_k u_k \underset{k\to\infty}{\rightharpoonup} nu \quad \text{in} \quad L^\infty_{w^*}(\mathbb{R}\times]0,\infty[). \tag{A.80}$$

Similarly, we get the convergences

$$n_k \bar{p}_k \underset{k\to\infty}{\rightharpoonup} n\bar{p} \quad \text{in} \quad L^\infty_{w^*}(\mathbb{R}\times]0,\infty[). \tag{A.81}$$

and

$$n_k(u_k+\bar{p}_k)u_k \underset{k\to\infty}{\rightharpoonup} qu \quad \text{in} \quad L^\infty_{w^*}(\mathbb{R}\times]0,\infty[), \tag{A.82}$$

with $q = nu + n\bar{p}$.

To complete the proof, we are going to show that $n^*(u_k)\bar{p}_k \underset{k\to\infty}{\longrightarrow} n^*(u)\bar{p}$ in $L^1(\mathbb{R}\times[0,T])$. From (A.78) and (A.79), we have

$$u_k(x,t) \underset{k\to\infty}{\longrightarrow} u(x,t),\ \bar{p}_k(x,t) \underset{k\to\infty}{\longrightarrow} \bar{p}(x,t) \quad \text{a.e. } (x,t)\in\mathbb{R}\times[0,T] \tag{A.83}$$

and there exists $h \in L^1(\mathbb{R} \times [0,T])$ such that for a subsequence, $|\bar{p}_k| \leqslant h$ a.e. Since n^* is continuous, we get

$$n^*(u_k)\bar{p}_k \underset{k\to\infty}{\longrightarrow} n^*(u)\bar{p} \;\text{ a.e., } \;\text{and } |n^*(u_k)\bar{p}_k| \leqslant n^*(0)h \in L^1(\mathbb{R} \times [0,T]), \tag{A.84}$$

then by dominated convergence

$$n^*(u_k)\bar{p}_k \underset{k\to\infty}{\longrightarrow} n^*(u)\bar{p} \;\text{ in }\; L^1(\mathbb{R} \times [0,T]). \tag{A.85}$$

Finally we get a solution of (A.35a)-(A.35c). Moreover, this solution satisfies also (A.73)-(A.74).

□

Remarks A.2. *We study here only the case where $p^0 = 0$. For the particular case of block initial data, we can nevertheless take any p^0 as in the corresponding section. The result can be extended to initial data such that on any interval $n = n^*$, the initial pressure is piecewise constant with 0 for the last constant, and 0 on other sets.*

A.4 The Riemann Problem analysis

A.4.1 The Riemann problem for the RMAR* model (A.26)-(A.27)

In this section, we briefly study the simple waves and then discuss the Riemann problem for the RMAR* system (A.26)-(A.27). And we refer the reader to [3] for more details on the derivation.

Simple waves for the RMAR* system (A.26)-(A.27)

Proposition A.6. *the solution of the RMAR* model (A.26)-(A.27) consists of either a wave of the first family (1-shock or 1-rarefaction) or a wave of the second family (2-contact discontinuity).*

Proof. (i) First characteristic field: We obtain the wave of the first family when a left state $U_l^\varepsilon = (n_l^\varepsilon, u_l^\varepsilon)$ is connected with a right state $U_r^\varepsilon = (n_r^\varepsilon, u_r^\varepsilon)$ through the curve

$$u_r^\varepsilon + \varepsilon p(n_r^\varepsilon, u_r^\varepsilon) = u_l^\varepsilon + \varepsilon p(n_l^\varepsilon, u_l^\varepsilon). \tag{A.86}$$

– If $u_r^\varepsilon < u_l^\varepsilon$, this wave (of the first family) is a 1-shock i.e. a jump discontinuity, travelling with the speed

$$\sigma^\varepsilon = \frac{n_r^\varepsilon u_r^\varepsilon - n_l^\varepsilon u_l^\varepsilon}{n_r^\varepsilon - n_l^\varepsilon}. \tag{A.87}$$

– On the other hand, if $u_r^\varepsilon > u_l^\varepsilon$, this wave of the first family is a 1-rarefaction i.e. a continuous solution of the form $(n^\varepsilon, u^\varepsilon)(\xi)$ (with $\xi = \frac{x}{t}$) given by

$$\begin{pmatrix} (n^\varepsilon)'(\xi) \\ (u^\varepsilon)'(\xi) \end{pmatrix} = \frac{r_1^\varepsilon(U^\varepsilon(\xi))}{\nabla\lambda_1^\varepsilon(U^\varepsilon(\xi)).r_1^\varepsilon(U^\varepsilon(\xi))}, \quad \lambda_1^\varepsilon(U_l^\varepsilon) \leqslant \xi \leqslant \lambda_1^\varepsilon(U_r^\varepsilon), \tag{A.88}$$

$$(n^\varepsilon, u^\varepsilon)(\xi) = \begin{cases} (n_l^\varepsilon, u_l^\varepsilon) & \text{for} \quad \xi < \lambda_1^\varepsilon(U_l^\varepsilon), \\ (n_r^\varepsilon, u_r^\varepsilon) & \text{for} \quad \xi > \lambda_2^\varepsilon(U_r^\varepsilon). \end{cases} \tag{A.89}$$

(ii) Second characteristic field: We obtain a wave of the second family i.e. a 2-contact discontinuity when $u_l^\varepsilon = u_r^\varepsilon$. In this case, this contact discontinuity between the left state $U_l^\varepsilon = (n_l^\varepsilon, u_r^\varepsilon)$ and the right state $U_r^\varepsilon = (n_r^\varepsilon, u_r^\varepsilon)$ travels with speed $u^\varepsilon = u_r^\varepsilon = u_l^\varepsilon$.

□

Solution to the Riemann problem for the RMAR* system (A.26)-(A.27)

Proposition A.7. *Let $U_l^\varepsilon = (n_l^\varepsilon, u_l^\varepsilon)$ and $U_r^\varepsilon = (n_r^\varepsilon, u_r^\varepsilon)$ be two given states respectively on the left and on the right. The general solution to the Riemann problem for the RMAR* model (A.26)-(A.27) consists of two simples waves separated by an intermediate state $\tilde{U}^\varepsilon = (\tilde{n}^\varepsilon, \tilde{u}^\varepsilon)$ which is the intersection point between the 1- wave curve through U_l^ε and the 2-contact discontinuity through U_r^ε.*

Proof. First, U_l^ε is connected with $\tilde{U}^\varepsilon$ through a wave of the first family (i.e. either 1-shock or 1-rarefaction according to the above discussion) and then $\tilde{U}^\varepsilon$ is connected with U_r^ε through a 2-contact discontinuity. Therefore,

$$\tilde{u}^\varepsilon + \varepsilon p(\tilde{n}^\varepsilon, \tilde{u}^\varepsilon) = u_l^\varepsilon + \varepsilon p(n_l^\varepsilon, u_l^\varepsilon) \quad \text{and} \quad \tilde{u}^\varepsilon = u_r^\varepsilon. \tag{A.90}$$

Hence, the density of the intermediate state $\tilde{U}^\varepsilon = (\tilde{n}^\varepsilon, \tilde{u}^\varepsilon = u_r^\varepsilon)$ is given by

$$\tilde{n}^\varepsilon = p(., u_r^\varepsilon)^{-1}[\varepsilon^{-1}(u_l^\varepsilon - u_r^\varepsilon) + p(n_l^\varepsilon, u_l^\varepsilon)]. \tag{A.91}$$

□

Remarks A.3. *If $u_r^\varepsilon > u_l^\varepsilon$, the equation (A.91) admits a solution if and only if $\varepsilon^{-1}(u_l^\varepsilon - u_r^\varepsilon) + p(n_l^\varepsilon, u_l^\varepsilon) > 0$. Otherwise i.e. if $u_l^\varepsilon + \varepsilon p(n_l^\varepsilon, u_l^\varepsilon) < u_r^\varepsilon$, then a vacuum state ($n = 0$) separates the two states U_l^ε and U_r^ε.*

Now let us discuss the solution to the Riemann problem in different cases.

(1) $u_r^\varepsilon < u_l^\varepsilon$. First, a 1-shock connects $U_l^\varepsilon = (n_l^\varepsilon, u_l^\varepsilon)$ to the intermediate state $\tilde{U}^\varepsilon = (\tilde{n}^\varepsilon, u_r^\varepsilon)$ and then a 2-contact discontinuity connects $\tilde{U}^\varepsilon$ to $U_r^\varepsilon = (n_r^\varepsilon, u_r^\varepsilon)$.

(2) $u_l^\varepsilon < u_r^\varepsilon < u_l^\varepsilon + \varepsilon p(n_l^\varepsilon, u_l^\varepsilon)$. The left state $U_l^\varepsilon = (n_l^\varepsilon, u_l^\varepsilon)$ is connected to the intermediate state $\tilde{U}^\varepsilon = (\tilde{n}^\varepsilon, u_r^\varepsilon)$ by a 1-rarefaction wave and then a 2-contact discontinuity connects $\tilde{U}^\varepsilon$ to the right state $U_r^\varepsilon = (n_r^\varepsilon, u_r^\varepsilon)$.

(3) $u_l^\varepsilon + \varepsilon p(n_l^\varepsilon, u_l^\varepsilon) < u_r^\varepsilon$. First a 1-rarefaction wave connects the left state $U_l^\varepsilon = (n_l^\varepsilon, u_l^\varepsilon)$ to the vacuum ($(0, \tilde{u}^\varepsilon)$ with $\tilde{u}^\varepsilon = u_l^\varepsilon + \varepsilon p(n_l^\varepsilon, u_l^\varepsilon)$) and then a 2-contact discontinuity connects the vacuum to the right state $U_r^\varepsilon = (n_r^\varepsilon, u_r^\varepsilon)$.

A.4.2 Simple waves for the SOMC system (A.29)-(A.31)

We recall that the SOMC system (A.29)-(A.31) is the formal limit of the RMAR* system (A.26)-(A.27) when $\varepsilon \longrightarrow 0$. As for the CPGD system (A.14)-(A.16), due to (A.31), $\bar{p}$ plays the role of a Lagrangian multiplier. Therefore one has to distinguish between the cases $n = n^*(u)$ and $n < n^*(u)$. Throughout the discussion below we consider the three quantities n, u and $\bar{p}$. Let $U_l = (n_l, u_l, \bar{p}_l)$ and $U_r = (n_r, u_r, \bar{p}_r)$ be two given states respectively on the left and on the right, such that

$$(n_l^\varepsilon, u_l^\varepsilon, \varepsilon p(n_l^\varepsilon, u_l^\varepsilon)) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} (n_l, u_l, \bar{p}_l)$$

and

$$(n_r^\varepsilon, u_r^\varepsilon, \varepsilon p(n_r^\varepsilon, u_r^\varepsilon)) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} (n_r, u_r, \bar{p}_r).$$

If $n_{l,r} = n^*(u_{l,r})$, then $\lim\limits_{\varepsilon \longrightarrow 0} \varepsilon p(n_{l,r}^\varepsilon, u_{l,r}^\varepsilon) = \bar{p}_{l,r}$ with $0 \leqslant \bar{p}_{l,r} < \infty$.

First characteristic field: 1-shocks.

Lemma A.8. *The 1-shock waves appear in the SOMC system only if $u_r < u_l$ and if (A.90) is satisfied.*

Proof. We shall distinguish the following cases.

1. $n_l < n^*(u_l), n_r = n^*(u_r)$ i.e. $\bar{p}_l = 0$ and $0 < \bar{p}_r < \infty$.
   In this case, when $\varepsilon \longrightarrow 0$, from (A.86) we have $u_l = u_r + \bar{p}_r > u_r$. Therefore, we have a 1-shock between the states $U_l = (n_l, u_l, \bar{p}_l = 0)$ and $U_r = (n^*(u_r), u_r, \bar{p}_r)$ travelling with a speed σ given by

$$\sigma = \frac{n^*(u_r)u_r - n_l u_l}{n^*(u_r) - n_l}. \tag{A.92}$$

   This situation models a "cluster growing" upstream: as soon as a faster vehicle catches up with the cluster, it adapts its velocity to the saturation density $n^*(u_r)$ and "is swallowed by" the cluster.

2. $n_l = n^*(u_l), n_r = n^*(u_r)$ i.e. $0 < \bar{p}_l < \infty$ and $0 < \bar{p}_r < \infty$.
   When $\bar{p}_r > \bar{p}_l$, from (A.86), we have

$$u_l + \bar{p}_l = u_r + \bar{p}_r \Longrightarrow u_l = u_r + \bar{p}_r - \bar{p}_l > u_r. \tag{A.93}$$

   Therefore we have a 1-shock travelling with the speed

$$\sigma = \frac{n^*(u_r)u_r - n^*(u_l)u_l}{n^*(u_r) - n^*(u_l)}. \tag{A.94}$$

   This situation models a "cluster slowing down" that leads to a merging of two clusters since the left cluster is faster than the right one. In contrast with the corresponding discussion in [6], here the wave speed σ is always finite. As soon as a collusion occurs, the velocity of the left cluster adjusts gradually to the right one through the shock wave. This propagation also involves the functional $\bar{p}_l$, see below.

□

First characteristic field: 1-rarefaction Waves.

Lemma A.9. *The 1-rarefaction waves appear in the SOMC system if $u_r > u_l$ and (A.90) is satisfied.*

Proof. Here also we shall distinguish two cases.

1. $n_l = n^*(u_l), n_r < n^*(u_r)$, therfore $0 < \bar{p}_l < \infty$ and $\bar{p}_r = 0$.
   When $\varepsilon \longrightarrow 0$, from (A.86) we have $u_r = u_l + \bar{p}_l > u_l$. This case describes a "cluster acceleration" leading to a "cluster growing" downstream. Indeed, the vehicles downstream are faster than the cluster: the cluster accelerates in order to reach its preferred velocity $u_l + \bar{p}_l = u_r$.

Here also, in contrast to [6], the velocity of the left state u_l changes gradually to u_r, therefore the functional $\bar{p}_l$ changes also gradually to $\tilde{p}_l \longrightarrow 0$.

2. $n_l = n^*(u_l), n_r = n^*(u_r)$, therefore $0 < \bar{p}_l < \infty$ and $0 < \bar{p}_r < \infty$.
   When $\bar{p}_l > \bar{p}_r$, from (A.86) we have

$$u_r + \bar{p}_r = u_l + \bar{p}_l \Longrightarrow u_r = u_l + \bar{p}_l - \bar{p}_r > u_l. \tag{A.95}$$

   This situation also models a "cluster acceleration" that leads to a merging of two clusters. The right cluster being faster than the left one, the left cluster accelerates and catches up with the right one since $u_r = u_l + \bar{p}_l - \bar{p}_r < u_l + \bar{p}_l$. Hence the left cluster adjusts gradually its velocity to the velocity u_r of the right one.

□

Second characteristic field: 2-contact discontinuities

Lemma A.10. *The 2-contact discontinuities appear in the SOMC system if* $u_r = u_l$.

Proof. We have the following cases.

1. $n_l < n^*(u_l), n_r < n^*(u_r)$, therefore $\bar{p}_l = \bar{p}_r = 0$,
2. $n_l = n^*(u_l), n_r = n^*(u_r)$, with $0 < \bar{p}_l = \bar{p}_r < \infty$.

In each of these two cases, at the limit $\varepsilon \longrightarrow 0$, in (A.86), we get $u_l = u_r = \tilde{u}$. Therefore, the solution consists of a 2-contact discontinuity travelling with velocity $\tilde{u}$ from $U_l = (n_l, \tilde{u}, \bar{p}_l)$ to $U_r = (n_r, \tilde{u}, \bar{p}_r)$.

□

A.4.3 Solution to the Riemann problem for the SOMC system (A.29)-(A.30)

In this subsection, we describe the solutions to the Riemann problem for the SOMC system (A.29)-(A.30), by combining the previously described elementary waves depending on whether $n = n^*(u)$ or $n < n^*(u)$.

Proposition A.11. *Let* $U_l = (n_l, u_l, \bar{p}_l)$ *and* $U_r = (n_r, u_r, \bar{p}_r)$ *be the initial data on the left and on the right, respectively. The solutions to the SOMC system (A.29)-(A.31) for these initial data consist of the following cases:*

(i) Case 1: $n_l < n^*(u_l), n_r < n^*(u_r)$ *i.e.* $\bar{p}_l = \bar{p}_r = 0$*;*
(ii) Case 2: $n_l = n^*(u_l), n_r < n^*(u_r)$ *i.e.* $0 \leqslant \bar{p}_l < \infty$ *and* $\bar{p}_r = 0$*;*
(iii) Case 3: $n_l < n^*(u_l), n_r = n^*(u_r)$ *i.e.* $\bar{p}_l = 0$ *and* $0 \leqslant \bar{p}_r < \infty$*;*
(iv) Case 4: $n_l = n^*(u_l), n_r = n^*(u_r)$ *i.e.* $0 < p_r < \infty, 0 < \bar{p}_l < \infty$.

Proof. **(i) Case 1:** $n_l < n^*(u_l), n_r < n^*(u_r)$ **i.e.** $\bar{p}_l = \bar{p}_r = 0$**.**

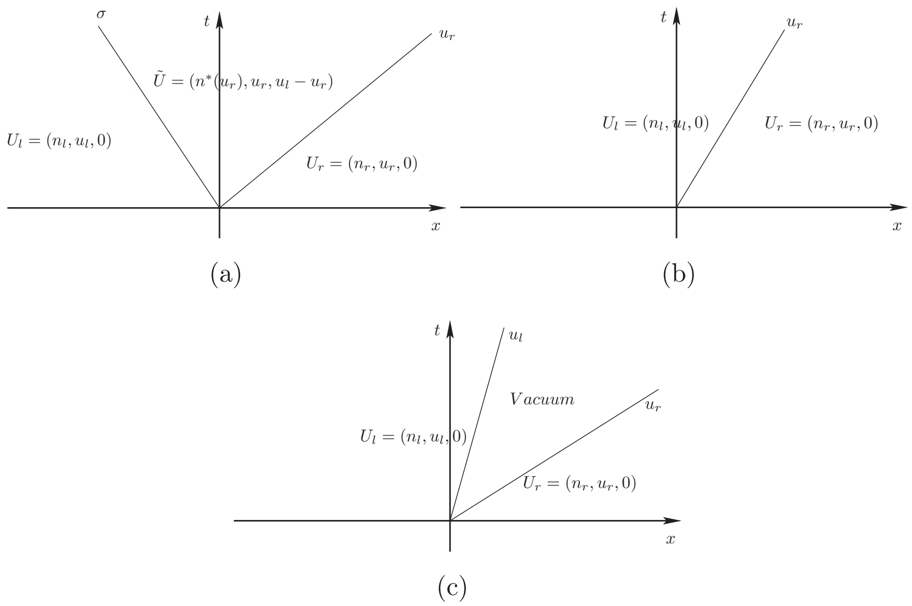


Figure A.4: (a): Subcase 1.1, (b): Subcase 1.2, (c): Subcase 1.3

(i-1) Subcase 1.1 $u_r < u_l$. In this case we have a cluster formation. In fact, the density of the intermediate state $\tilde{n}$ increases and tends to $n^*(u_r)$. At the same time, $\varepsilon p(\tilde{n}, u_r) \longrightarrow \tilde{p} = u_l - u_r$. Therefore the intermediate state that characterizes the cluster is given by $\tilde{U} = (n^*(u_r), u_r, \tilde{p} = u_l - u_r)$. This intermediate state is separated from the left state $U_l = (n_l, u_l, \bar{p}_l = 0)$ and the right state $U_r = (n_r, u_r, \bar{p}_r = 0)$ respectively by a 1-shock travelling with velocity σ and a contact discontinuity with velocity u_r. The schock speed σ is given by

$$\sigma = \frac{n_r u_r - n_l u_l}{n_r - n_l}.$$

This situation is illustrated by Figure A.4 (a).

(i-2) Subcase 1.2 $u_l < u_r < u_l + p_l$, therefore $u_l = u_r$. This case is solved by a single contact discontinuity travelling with the velocity $\tilde{u} = u_l = u_r$ that connects

$U_l = (n_l, u_l, \bar{p}_l = 0)$ with $U_r = (n_r, u_r, \bar{p}_r = 0)$. An example is shown in Figure A.4 (b).

(i-3) Subcase 1.3 $u_l + p_l < u_r$ *i.e.* $u_l < u_r$. In this situation the vacuum appears. It is separated from the left state $U_l = (n_l, u_l, \bar{p}_l = 0)$ by a contact discontinuity (with velocity u_l) and another contact discontinuity (with velocity u_r) connects the vacuum with the right state $U_r = (n_r, u_r, \bar{p}_r = 0)$, see Figure A.4 (c) for illustration.

(ii) Case 2: $n_l = n^*(u_l), n_r < n^*(u_r)$ **i.e.** $0 \leqslant \bar{p}_l < \infty$ **and** $\bar{p}_r = 0$.

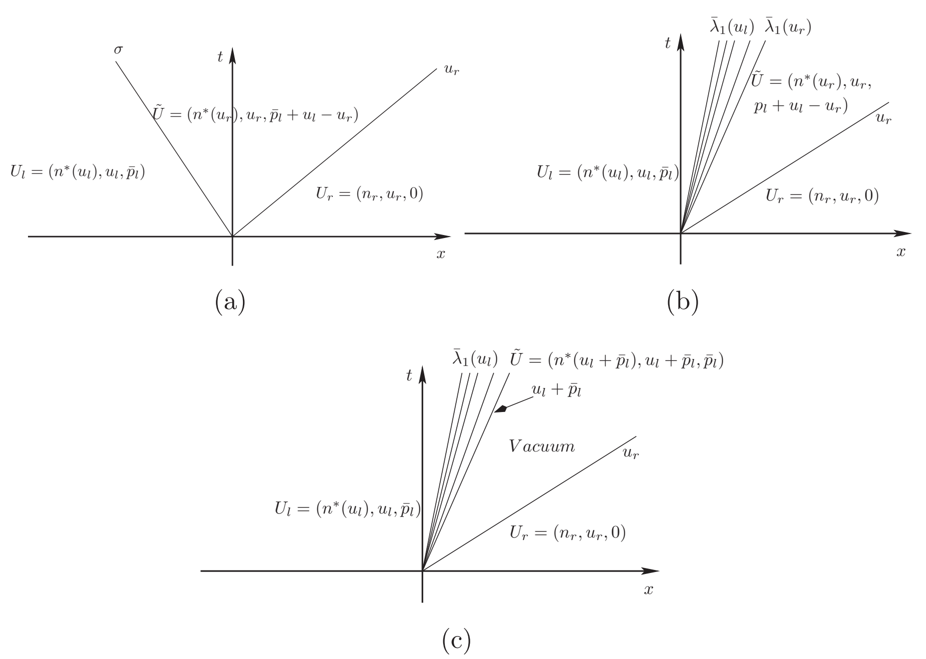


Figure A.5: (a): Subcase 2.1, (b): Subcase 2.2, (c): Subcase 2.3

(ii-1) Subcase 2.1 $u_r < u_l$. Here we have a "cluster growing" downstream. The cluster being faster than the vehicles ahead, must adapt its velocity gradually to u_r through a 1-shock connecting u_r to the intermediate state $\tilde{U} = (n^*(u_r), u_r, \tilde{p} = \bar{p}_l + u_l - u_r)$. This is illustrated by Figure A.5 (a).

(ii-2) Subcase 2.2 $u_l < u_r < u_l + \bar{p}_l$. The vehicles ahead of the cluster are faster than this one. However, their velocity u_r is less than the cluster preferred velocity $u_l + \bar{p}_l$ therefore, we have a "cluster acceleration" that leads to a "cluster growing" downstream. The solution is of the following form: The left state $U_l = (n^*(u_l), u_l, \bar{p}_l)$ is connected to the intermediate state $\tilde{U} = (n^*(u_r), u_r, \tilde{p} = \bar{p}_l + u_l - u_r)$ with a 1-rarefaction wave, then $\tilde{U}$ is connected to $U_r = (n_r, u_r, \bar{p}_r = 0)$ with a contact discontinuity of velocity u_r. This is illustrated in Figure A.5 (b).

(ii-3) Subcase 2.3: $u_l + \bar{p}_l < u_r$. Here the velocity u_r of the vehicles ahead of the cluster is greater than the cluster preferred velocity $u_l + \bar{p}_l$. Therefore we have a "cluster acceleration" in order to reach the preferred velocity $u_l + \bar{p}_l$. But, a vacuum appears since $u_r > u_l + \bar{p}_l$. The solution is as follows: The left state $U_l = (n^*(u_l), u_l, \bar{p}_l)$ is connected to the intermediate state $\tilde{U} = (n^*(u_l+\bar{p}_l), u_l+\bar{p}_l, \bar{p}_l)$ through a 1-rarefaction wave. Then $\tilde{U}$ is connected to the vacuum by a 2-contact discontinuity. Then the vacuum is connected with U_r by a 2-contact discontinuity. An example is described in Figure A.5 (c).

(iii) Case 3: $n_l < n^*(u_l), n_r = n^*(u_r)$ **i.e.** $\bar{p}_l = 0$ **and** $0 \leqslant \bar{p}_r < \infty$**.**

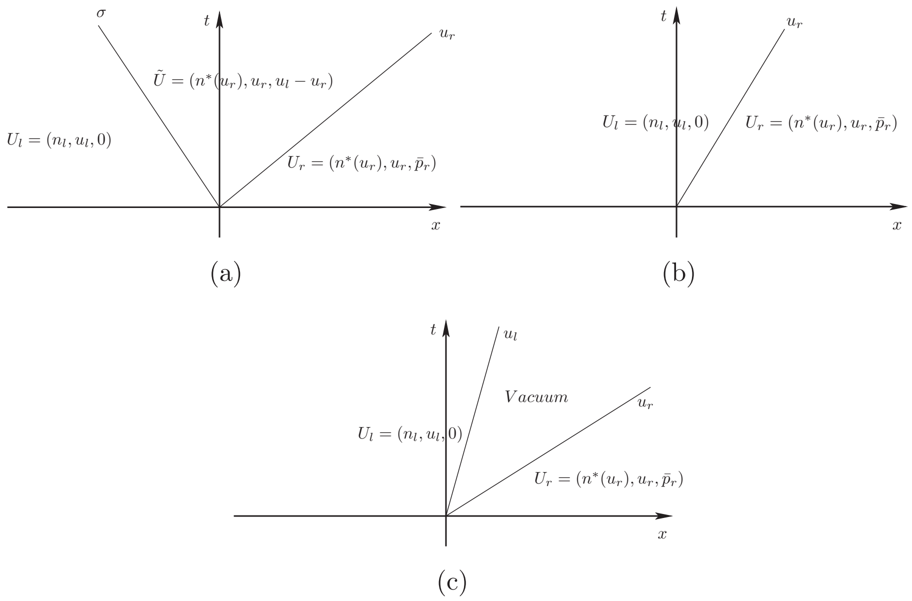


Figure A.6: (a): Subcase 3.1, (b): Subcase 3.2, (c): Subcase 3.3

(iii-1) Subcase 3.1 $u_r < u_l$. In this situation we have a "cluster growing" upstream. The vehicles behind the cluster are faster that this one. As soon as a vehicle catches up with the cluster, it slows down, adapts its velocity to the saturation dentity $n^*(u_r)$ through a 1-shock connecting the left state U_l to an intermediate state $\tilde{U}$ and becomes a part of (or "is swallowed by") the cluster. The solution is quasi similar to that of Subcase 1.1 and the only difference is that here $p_r \neq 0$. See Figure A.6 (a).

(iii-2) Subcase 3.2 $u_l < u_r < u_l + \bar{p}_l = u_l$, therefore $u_l = u_r$. Like in the Subcase 1.2, here also the solution consists of a single contact discontinuity with velocity $\tilde{u} = u_l = u_r$, connecting the left state $U_l = (n_l, u_l, \bar{p}_l = 0)$ and the right state $U_r = (n^*(u_r), u_r, \bar{p}_r)$. See Figure A.6 (b).

(iii-3) Subcase 3.3 $u_l + \bar{p}_l < u_r$, therefore $u_l < u_r$. The downstream cluster being faster than the vehicles behind, a vacuum state appears between them. Since $\bar{p}_l = 0$, the left state $U_l = (n_l, u_l, \bar{p}_l = 0)$ is connected to the vacuum with a contact discontinuity of velocity u_l. Then the vacuum is separated from the right state $U_r = (n^*(u_r), u_r, \bar{p}_r)$ with another contact discontinuity of velocity u_r. See (c) of Figure A.6 for illustration.

(iv) Case 4: $n_l = n^*(u_l), n_r = n^*(u_r)$ **i.e.** $0 < p_r < \infty, 0 < \bar{p}_l < \infty$.

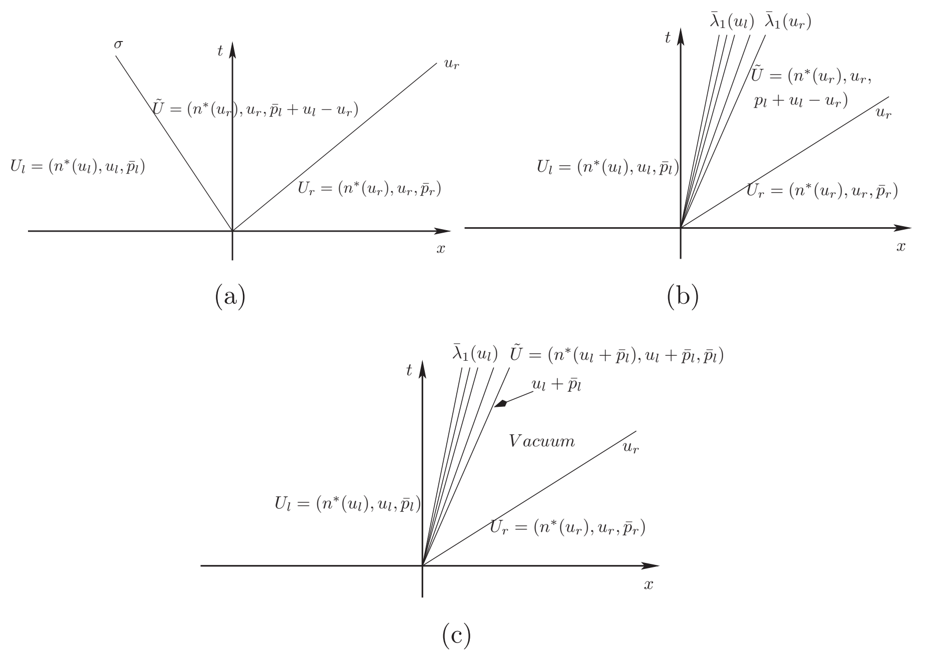


Figure A.7: (a): Subcase 4.1, (b): Subcase 4.2, (c): Subcase 4.3

(iv-1) Subcase 4.1 $u_r < u_l$. Here we have a "cluster slowing down": a 1-shock leading to a merging of two clusters. The left cluster is faster than the right one. When the two clusters meet, the left one slows down and adapts gradually its velocity to the velocity u_r of the right one. The solution is therefore almost similar to the one of Subcase 1.1 except that here the intermediate state is now given by $\tilde{U} = (n^*(u_r), u_r, \tilde{p} = \bar{p}_l + u_l - u_r)$, with here $p_l \neq 0$ and $p_r \neq 0$. See Figure A.7 (a) for illustration.

(iv-2) Subcase 4.2 $u_l < u_r < u_l + p_l$. In this situation we have a "cluster acceleration" leading to the merging of two clusters. The right cluster is faster than the left one, but its velocity u_r is less than the preferred velocity $u_l + \bar{p}_l$ of the left cluster, which then accelerates and gradually adapts its velocity to u_r. The solution

is almost similar to the one of Subcase 2.2, except that here $p_r \neq 0$. An example is given by Figure A.7 (b).

(iv-3) Subcase 4.3 $u_l + \bar{p}_l < u_r$. The velocity u_r of the right cluster is larger than the preferred velocity of the left cluster $u_l + \bar{p}_l$. Therefore the left cluster accelerates to reach its preferred velocity. However, the two clusters do not collide since $u_r > u_l + \bar{p}_l$, so that a vacuum state appears between them, as in Subcase 2.3 (with here $p_r \neq 0$). We have illustrated this situation in Figure A.7 (c).

□

A.5 Concluding remarks

The model presented in this paper, contrarily to [6], takes into account the fact that the maximal density depends on the velocity. Furthermore, the proposed model behaves as the Lighthill & Whitham model, [47] when the maximal density constraint is saturated, and on the other hand, in the free flow regime, it becomes a pressureless gas model. This double-sided behaviour has been highlighted in the analysis of the Riemann problem. We have proved an existence result of weak solution for the model and discussed the associated Riemann problem. This work is motivated by the fact that in practice a correlation exists between the maximal density constraint and the velocity. The approach in this paper opens many perspectives, and futur research can be carried out towards several directions. First, this model is designed on a single highway framework. A further interesting issue is to extend the model to the case of multilanes highways with overtaking possibilities. Also an extension to road networks and a comparison with other traffic models would be worthwhile.

Appendix

Proof. (**of Lemma A.1**)

We have

$$\nabla \lambda_1 . r_1 = -2\partial_n p - n \begin{pmatrix} 1 \\ \\ \frac{-\partial_n p}{1+\partial_u p} \end{pmatrix}^t \begin{pmatrix} \partial_{nn} p & \partial_{un} p \\ \\ \partial_{un} p & \partial_{uu} p \end{pmatrix} \begin{pmatrix} 1 \\ \\ \frac{-\partial_n p}{1+\partial_u p} \end{pmatrix}. \tag{A.96}$$

As $\partial_n p \geqslant 0$, the first term at the right hand side of the equation (A.96) is non positive.

Let us denote by $H(n,u)$ the Hessian matrix of p $\begin{pmatrix} \partial_{nn}p & \partial_{un}p \\ \partial_{un}p & \partial_{uu}p \end{pmatrix}$.

Rewriting, $H(n,u)$ in terms of (n,n^*) and the derivative of n^*, we obtain,

$$H(n,u) = \begin{pmatrix} \partial_{nn}p^\dagger & \partial_{nn^*}p^\dagger \frac{\mathrm{d}n^*(u)}{\mathrm{d}u} \\ \partial_{nn^*}p^\dagger \frac{\mathrm{d}n^*(u)}{\mathrm{d}u} & \partial_{n^*n^*}p^\dagger \left(\frac{\mathrm{d}n^*(u)}{\mathrm{d}u}\right)^2 + \partial_{n^*}p^\dagger \frac{\mathrm{d}^2n^*(u)}{\mathrm{d}u^2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{\mathrm{d}n^*(u)}{\mathrm{d}u} \end{pmatrix} \begin{pmatrix} \partial_{nn}p^\dagger & \partial_{nn^*}p^\dagger \\ \partial_{nn^*}p^\dagger & \partial_{n^*n^*}p^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\mathrm{d}n^*(u)}{\mathrm{d}u} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & \partial_{n^*}p^\dagger \frac{\mathrm{d}^2n^*(u)}{\mathrm{d}u^2} \end{pmatrix}.$$

with

$$p^\dagger(n,n^*) = \left(\frac{1}{n} - \frac{1}{n^*}\right)^{-\gamma} \quad \text{and } \gamma > 0. \tag{A.97}$$

Let us denote by $\tilde{H}(n,n^*)$ the matrix $\begin{pmatrix} \partial_{nn}p^\dagger & \partial_{nn^*}p^\dagger \\ \partial_{nn^*}p^\dagger & \partial_{n^*n^*}p^\dagger \end{pmatrix}$.

In order to show that $\nabla\lambda_1 \cdot r_1$ keeps a constant sign ($\nabla\lambda_1 \cdot r_1 < 0$) we are looking for a condition such that $H(n,n^*)$ is positive definite. Since $\forall\ \gamma > 0$, we have

$$\partial_n p^\dagger = \frac{\gamma}{n^2 Z^{\gamma+1}} > 0;\ \partial_{n^*}p^\dagger = \frac{-\gamma}{(n^*)^2 Z^{\gamma+1}} < 0;\ \partial_{nn^*}p^\dagger = \frac{-\gamma(\gamma+1)}{n^2(n^*)^2 Z^{\gamma+2}} < 0;$$

$$\partial_{nn}p^\dagger = \frac{-2\gamma}{n^3 Z^{\gamma+1}} + \frac{\gamma(\gamma+1)}{n^4 Z^{\gamma+2}} > 0;\quad \partial_{n^*n^*}p^\dagger = \frac{2\gamma}{(n^*)^3 Z^{\gamma+1}} + \frac{\gamma(\gamma+1)}{(n^*)^4 Z^{\gamma+2}} > 0;$$

$$\text{with} \quad Z = \left(\frac{1}{n} - \frac{1}{n^*}\right),$$

Then the Hessian matrix of $\tilde{p}^{\dagger}(n,n^*)$ is given by

$$\tilde{H}(n,n^*) = \begin{pmatrix} \frac{-2\gamma}{n^3 Z^{\gamma+1}} + \frac{\gamma(\gamma+1)}{n^4 Z^{\gamma+2}} & -\frac{\gamma(\gamma+1)}{n^2 n^{*2} Z^{\gamma+2}} \\ -\frac{\gamma(\gamma+1)}{n^2 n^{*2} Z^{\gamma+2}} & \frac{2\gamma}{n^{*3} Z^{\gamma+1}} + \frac{\gamma(\gamma+1)}{n^{*4} Z^{\gamma+2}} \end{pmatrix}$$

and its determinant is

$$\det(\tilde{H}(n,n^*)) = \frac{2\gamma}{n^3 n^{*3} Z^{\gamma+1}} \left(\frac{\gamma(\gamma+1)}{Z^{\gamma+1}} - \frac{2\gamma}{Z^{\gamma+1}} \right).$$

For all $\gamma > 1$, $\det(\tilde{H}(n,n^*)) > 0$, therefore $\tilde{H}(n,u)$ is positive definite. Then, $H(n,u)$ is positive definite if

$$(\partial_n p)^2 \, \partial_{n^*} p \, \frac{\mathrm{d}^2 n^*}{\mathrm{d}u^2} \geqslant 0. \tag{A.98}$$

The inequality (A.98) is satisfied since $(\partial_n p)^2 \geqslant 0$, $\partial_{n^*} p \leqslant 0$ and due to the assumption **(A-3)**, $\dfrac{\mathrm{d}^2 n^*}{\mathrm{d}u^2} \leqslant 0$. Therefore, the eigenvalue λ_1 is genuinely non linear.

□

Bibliographie

[1] Chaouki Abourjaily and Philippe Bénilan. Symmetrization of quasi-linear parabolic problems. *Rev. Un. Mat. Argentina*, 41(1) : pp. 1–13, 1998. Dedicated to the memory of Julio E. Bouillet.

[2] A. R. A. Anderson. A hybrid mathematical model of solid tumour invasion: The importance of cell adhesion. *Math. Med. Biol. IMA J.* 22: pp. 163–186, 2005.

[3] A. Aw and Michel Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.* 60(3) : pp. 916–938 (electronic), 2000.

[4] Masako Bando, Katsuya Hasebe, Akihiro Nakayama, A. Shibata, and Yuki Sugiyama. Dynamical model of traffic congestion and numerical simulation. *Phys. Rev. E*, 51: pp. 1035–1042, 1995.

[5] Florent Berthelin. Existence and weak stability for a pressureless model with unilateral constraint. *Math. Models Methods Appl. Sci.* 12(2) : pp. 249–272, 2002.

[6] Florent Berthelin, Pierre Degond, Marcello Delitala, and Michel Rascle. A model for the formation and evolution of traffic jams. *Arch. Ration. Mech. Anal.* 187(2) : pp. 185–220, 2008.

[7] Florent Berthelin, Pierre Degond, Valérie Le Blanc, Salissou Moutari, Michel Rascle, and Julien Royer. A traffic-flow model with constraints for the modeling of traffic jams. *Math. Models Methods Appl. Sci.* 18(suppl.) : pp. 1269–1298, 2008.

[8] Piotr Biler. Local and global solvability of some parabolic systems modelling chemotaxis. *Adv. Math. Sci. Appl.* 8(2) : pp. 715–743, 1998.

[9] Piotr Biler, Lucilla Corrias, and Jean Dolbeault. Large mass self-similar solutions of the parabolic-parabolic Keller–Segel model of chemotaxis: 2009.

[10] Adrien BLANCHET, Jean DOLBEAULT, and Benoît PERTHAME. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations*, : No. 44, 32 pp. (electronic), 2006.

[11] Yann BRENIER and Emmanuel GRENIER. Sticky particles and scalar conservation laws. *SIAM J. Numer. Anal.* 35(6) : 2317–2328 (electronic), 1998.

[12] Alberto BRESSAN and Tong YANG. On the convergence rate of vanishing viscosity approximations. *Comm. Pure Appl. Math.* 57(8) : pp. 1075–1109, 2004.

[13] Elena O. BUDRENE and Howard C. BERG. Complex patterns formed by motile cells of escherichia coli. *Nature*, 349: pp. 630–633, 1991.

[14] Elena O. BUDRENE and Howard C. BERG. Dynamics of formation of symmetrical patterns by chemotactic bacteria. *Nature*, 376: pp. 49–53, 1995.

[15] Vincent CALVEZ and Lucilla CORRIAS. The parabolic-parabolic Keller-Segel model in $\mathbb{R}^2$. *Commun. Math. Sci.* 6(2) : pp. 417–447, 2008.

[16] Alina CHERTOCK and Alexander KURGANOV. A second-order positivity preserving central-upwind scheme for chemotaxis and haptotaxis models. *Numer. Math.* 111(2) : pp. 169–205, 2008.

[17] Rinaldo M. COLOMBO. A 2×2 hyperbolic traffic flow model. *Math. Comput. Modelling*, 35(5-6) : pp. 683–688, 2002. Traffic flow—modelling and simulation.

[18] Constantine M. DAFERMOS. *Hyperbolic conservation laws in continuum physics*, vol. 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 2000.

[19] Anne-Laure DALIBARD. Homogenization of a quasilinear parabolic equation with vanishing viscosity. *J. Math. Pures Appl. (9)*, 86(2) : pp. 133–154, 2006.

[20] Anne-Laure DALIBARD. Stability of periodic stationary solutions of scalar conservation laws with space-periodic flux. *Preprint*, : pp. 1–37, 2009.

[21] Pierre DEGOND and Marcello DELITALA. Modelling and simulation of vehicular traffic jam formation. *Kinet. Relat. Models*, 1(2) : pp. 279–293, 2008.

[22] Marcello DELITALA and Andrea TOSIN. Mathematical modeling of vehicular traffic: a discrete kinetic theory approach. *Math. Models Methods Appl. Sci.* 17(6) : pp. 901–932, 2007.

[23] Ronald J. DIPERNA. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.* 91(1) : pp. 1–30, 1983.

[24] Robert F. DRESSLER. Mathematical solution of the problem of roll-waves in inclined open channels. *Comm. Pure Appl. Math.* 2: pp. 149–194, 1949.

[25] Yekaterina EPSHTEYN and Alexander KURGANOV. New interior penalty discontinuous Galerkin methods for the Keller-Segel chemotaxis model. *SIAM J. Numer. Anal.* 47(1) : pp. 386–408, 2008/09.

[26] Paul C. FIFE. *Dynamics of internal layers and diffusive interfaces*, vol. 53 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

[27] Francis FILBET. A finite volume scheme for the Patlak-Keller-Segel chemotaxis model. *Numer. Math.* 104(4) : pp. 457–488, 2006.

[28] Heinrich FREISTÜHLER and Denis SERRE. L^1-stability of shock waves in scalar viscous conservation laws. *Comm. Pure Appl. Math.* 51(3) : pp. 291–301, 1998.

[29] Herbert GAJEWSKI and Klaus ZACHARIAS. Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.* 195: pp. 77–114, 1998.

[30] R. A. GARDNER. On the structure of the spectra of periodic travelling waves. *J. Math. Pures Appl. (9)*, 72(5) : pp. 415–439, 1993.

[31] Denos C. GAZIS, Robert HERMAN, and Richard W. ROTHERY. Nonlinear follow-the-leader models of traffic flow. *Operations Res.* 9: pp. 545–567, 1961.

[32] Jonathan GOODMAN and Zhou Ping XIN. Viscous limits for piecewise smooth solutions to systems of conservation laws. *Arch. Rational Mech. Anal.* 121(3) : pp. 235–265, 1992.

[33] J. M. GREENBERG. Extensions and amplifications of a traffic model of Aw and Rascle. *SIAM J. Appl. Math.* 62(3) : 729–745 (electronic), 2001/02.

[34] Emmanuel GRENIER and Frédéric ROUSSET. Stability of one-dimensional boundary layers by using Green's functions. *Comm. Pure Appl. Math.* 54(11) : pp. 1343–1385, 2001.

[35] Olivier GUÈS, Guy MÉTIVIER, Mark WILLIAMS, and Kevin ZUMBRUN. Existence and stability of multidimensional shock fronts in the vanishing viscosity limit. *Arch. Ration. Mech. Anal.* 175(2) : pp. 151–244, 2005.

[36] Dirk HELBING. Improved fluid-dynamic model for vehicular traffic. *Phys. Rev. E*, 51: pp. 3164–3169, 1995.

[37] Daniel HENRY. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1981.

[38] Robert HERMAN and Ilya PRIGOGINE. *Kinetic theory of vehicular traffic*, of. American Elsevier, 1971.

[39] Dirk HORSTMANN. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.* 105(3) : pp. 103–165, 2003.

[40] Reinhard ILLNER, Axel KLAR, and Thorsten MATERNE. Vlasov-Fokker-Planck models for multilane traffic flow. *Commun. Math. Sci.* 1(1) : pp. 1–12, 2003.

[41] Evelyn F. KELLER and Lee A. SEGEL. Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.* 26: pp. 399–415, 1970.

[42] Axel KLAR, Reinhart D. KÜHNE, and Raimund WEGENER. Mathematical models for vehicular traffic. *Surveys Math. Indust.* 6(3) : pp. 215–239, 1996.

[43] Axel KLAR and Raimund WEGENER. Enskog-like kinetic models for vehicular traffic. *J. Statist. Phys.* 87(1-2) : pp. 91–114, 1997.

[44] Nancy KOPELL and L. N. HOWARD. Bifurcations and trajectories joining critical points. *Advances in Math.* 18(3) : pp. 306–358, 1975.

[45] Peter D. LAX. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, of. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[46] Valérie LE BLANC. L^1-stability of periodic stationary solutions of scalar convection-diffusion equations. *J. Differential Equations*, 247(6) : pp. 1746–1761, 2009.

[47] Michael James LIGHTHILL and G. B. WHITHAM. On kinematic waves. I. Flood movement in long rivers. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.* 229: pp. 281–345, 1955.

[48] Americo MARROCCO. Numerical simulation of chemotactic bacteria aggregation via mixed finite elements. *M2AN Math. Model. Numer. Anal.* 37(4) : pp. 617–630, 2003.

[49] Hiroshi MATANO. Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 29(2) : pp. 401–441, 1982.

[50] Toshitaka NAGAI, Takasi SENBA, and Kiyoshi YOSHIDA. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.* 40(3) : pp. 411–433, 1997.

[51] Kai NAGEL and Michael SCHRECKENBERG. A cellular automaton model for freeway traffic. *J. Phys.* 2: pp. 2221–2229, 1992.

[52] Paul NELSON. A kinetic model of vehicular traffic and its associated bimodal equilibrium solutions. In *Special issue devoted to the Proceedings of the 13th International Conference on Transport Theory (Riccione, 1993)*, vol. 24. 1-3, pp. 383–409 1995.

[53] Pascal NOBLE. Persistence of roll waves for the Saint Venant equations. *SIAM J. Math. Anal.* 40(5) : pp. 1783–1814, 2008/09.

[54] Pascal NOBLE. Roll-waves in general hyperbolic systems with source terms. *SIAM J. Appl. Math.* 67(4) : 1202–1212 (electronic), 2007.

[55] Myunghyun OH and Kevin ZUMBRUN. Low-frequency stability analysis of periodic traveling-wave solutions of viscous conservation laws in several dimensions. *Z. Anal. Anwend.* 25(1) : pp. 1–21, 2006.

[56] Myunghyun OH and Kevin ZUMBRUN. Stability and asymptotic behavior of periodic traveling wave solutions of viscous conservation laws in several dimensions. *Preprint*, : pp. 1–19, 2008.

[57] Myunghyun OH and Kevin ZUMBRUN. Stability of periodic solutions of conservation laws with viscosity: analysis of the Evans function. *Arch. Ration. Mech. Anal.* 166(2) : pp. 99–166, 2003.

[58] Myunghyun OH and Kevin ZUMBRUN. Stability of periodic solutions of conservation laws with viscosity: pointwise bounds on the Green function. *Arch. Ration. Mech. Anal.* 166(2) : pp. 167–196, 2003.

[59] Clifford S. PATLAK. Random walk with persistence and external bias. *Bull. Math. Biophys.* 15: pp. 311–338, 1953.

[60] S. L. PAVERI-FONTANA. On Boltzmann-like treatments for traffic flow. *Trans. Res.* 9: pp. 225–235, 1975.

[61] H. J. PAYNE. FREFLO: A macroscopic simulation model of freeway traffic. *Trans. Res. Rec.* 722: pp. 68–75, 1979.

[62] Paul I. RICHARDS. Shock waves on the highway. *Operations Res.* 4: pp. 42–51, 1956.

[63] Frédéric ROUSSET. Viscous approximation of strong shocks of systems of conservation laws. *SIAM J. Math. Anal.* 35(2) : 492–519 (electronic), 2003.

[64] Norikazu SAITO. Conservative upwind finite-element method for a simplified Keller-Segel system modelling chemotaxis. *IMA J. Numer. Anal.* 27(2) : pp. 332–365, 2007.

[65] Denis SERRE. L^1-stability of nonlinear waves in scalar conservation laws. In *Evolutionary equations. Vol. I*, Handb. Differ. Equ., pp. 473–553: North-Holland, Amsterdam, 2004.

[66] Jacques SIMON. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146: pp. 65–96, 1987.

[67] Michael E. TAYLOR. *Partial differential equations. III*, vol. 117 of *Applied Mathematical Sciences*, Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.

[68] Blake TEMPLE. System of conservation laws with coinciding shock and rarefaction curves. *Contemp. Math.* 17: pp. 143–151, 1983.

[69] Rebecca TYSON, S. R. LUBKIN, and J. D. MURRAY. Model and analysis of chemotactic bacterial patterns in a liquid medium. *J. Math. Biol.* 38(4) : pp. 359–375, 1999.

[70] Rebecca TYSON, L. G. STERN, and Randall J. LEVEQUE. Fractional step methods applied to a chemotaxis model. *J. Math. Biol.* 41(5) : pp. 455–475, 2000.

[71] Aizik Isaakovich VOL'PERT. Spaces BV and quasilinear equations. *Mat. Sb. (N.S.)* 73 (115): pp. 255–302, 1967.

[72] Christoph WALKER and Glenn F. WEBB. Global existence of classical solutions for a haptotaxis model. *SIAM J. Math. Anal.* 38(5) : 1694–1713 (electronic), 2006/07.

[73] H. M. ZHANG. A non-equilibrium traffic model devoid of gas-like behaviour. *Trans. Res. B.* 36: pp. 275–298, 2002.

[74] Kevin ZUMBRUN and Peter HOWARD. Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.* 47(3) : pp. 741–871, 1998.

[75] Kevin ZUMBRUN and Denis SERRE. Viscous and inviscid stability of multidimensional planar shock fronts. *Indiana Univ. Math. J.* 48(3) : pp. 937–992, 1999.